PROBLEMS IN REAL ANALYSIS

Second Edition

A Workbook with Solutions

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FOREWORD

This book contains complete solutions to the 609 problems in the third edition of *Principles of Real Analysis*, Academic Press, 1998. The problems have been spread over forty sections which follow the format of the book.

All solutions are based on the material covered in the text with frequent references to the results in the text. For instance, a reference to Theorem 7.3 refers to Theorem 7.3 and a reference to Example 28.4 refers to Example 28.4, both in the third edition of *Principles of Real Analysis*.

This problem book will be beneficial to students only if they use it "properly." That is to say, if students look at a solution of a problem *only after* trying very hard to solve the problem. Students will do themselves great injustice by reading a solution without any prior attempt on the problem. It should be a real challenge to students to produce solutions which are different from the ones presented here.

We would like to express our most sincere thanks to all the people who made constructive recommendations and corrections regarding the text and the problems. Special thanks are due to Professor Yuri Abramovich for his contributions and suggestions during the writing of this problem book.

C. D. ALIPRANTIS AND O. BURKINSHAW
West Lafayette, Indiana
July, 1998

CHAPTER 1

FUNDAMENTALS OF REAL ANALYSIS

1. ELEMENTARY SET THEORY

Problem 1.1. *Establish the following set theoretic relations:*

1. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ *and* $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;
2. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ *and* $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$;
3. $A \setminus B = A \cap B^c$;
4. $A \subseteq B \iff B^c \subseteq A^c$; *and*
5. $(A \cup B)^c = A^c \cap B^c$ *and* $(A \cap B)^c = A^c \cup B^c$.

Also, for an arbitrary function $f\colon X \to Y$, *establish the following claims:*

6. $f\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} f(A_i)$;
7. $f\left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I} f(A_i)$;
8. $f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i)$;
9. $f^{-1}\left(\bigcap_{i \in I} B_i\right) = \bigcap_{i \in I} f^{-1}(B_i)$; *and*
10. $f^{-1}(B^c) = \left[f^{-1}(B)\right]^c$

Solution. (1) We establish the first formula only. We have

$$\begin{aligned} x \in (A \cup B) \cap C &\iff x \in A \cup B \text{ and } x \in C \\ &\iff [x \in A \text{ or } x \in B] \text{ and } x \in C \\ &\iff [x \in A \text{ and } x \in C] \text{ or } [x \in B \text{ and } x \in C] \\ &\iff x \in A \cap C \text{ or } x \in B \cap C \\ &\iff x \in (A \cap C) \cup (B \cap C). \end{aligned}$$

(2) Again, we establish the first formula only. Observe that

$$x \in (A \cup B) \setminus C \iff x \in A \cup B \text{ and } x \notin C$$

$$\begin{aligned}
&\iff [x \in A \text{ or } x \in B] \text{ and } x \notin C \\
&\iff [x \in A \text{ and } x \notin C] \text{ or } [x \in B \text{ and } x \notin C] \\
&\iff x \in A \setminus C \text{ or } x \in B \setminus C \\
&\iff x \in (A \setminus C) \cup (B \setminus C).
\end{aligned}$$

(3) Note that

$$\begin{aligned}
x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\
&\iff x \in A \text{ and } x \in B^c \iff x \in A \cap B^c.
\end{aligned}$$

(4) Let $A \subseteq B$. Then, $x \in B^c$ implies $x \notin B$ and so $x \notin A$ (i.e., $x \in A^c$) so that $B^c \subseteq A^c$. On the other hand, if $B^c \subseteq A^c$ holds, then (by the preceding case) we have $A = (A^c)^c \subseteq (B^c)^c = B$.

(5) Note that

$$\begin{aligned}
x \in (A \cap B)^c &\iff x \notin A \cap B \iff x \notin A \text{ or } x \notin B \\
&\iff x \in A^c \text{ or } x \in B^c \iff x \in A^c \cup B^c.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x \in (A \cup B)^c &\iff x \notin A \cup B \iff x \notin A \text{ and } x \notin B \\
&\iff x \in A^c \text{ and } x \in B^c \iff x \in A^c \cap B^c.
\end{aligned}$$

(6) We have

$$\begin{aligned}
y \in f\Big(\bigcup_{i \in I} A_i\Big) &\iff \exists\ x \in \bigcup_{i \in I} A_i \text{ with } y = f(x) \\
&\iff \exists\ i \in I \text{ with } x \in A_i \text{ and } y = f(x) \\
&\iff \exists\ i \in I \text{ with } y \in f(A_i) \iff y \in \bigcup_{i \in I} f(A_i).
\end{aligned}$$

(7) From the inclusion $f\big(\bigcap_{i \in I} A_i\big) \subseteq f(A_j)$ for each j, we see that

$$f\Big(\bigcap_{i \in I} A_i\Big) \subseteq \bigcap_{i \in I} f(A_i).$$

(8) We have

$$\begin{aligned}
x \in f^{-1}\Big(\bigcup_{i \in I} B_i\Big) &\iff f(x) \in \bigcup_{i \in I} B_i \iff \exists\ i \in I \text{ with } f(x) \in B_i \\
&\iff \exists\ i \in I \text{ with } x \in f^{-1}(B_i) \iff x \in \bigcup_{i \in I} f^{-1}(B_i).
\end{aligned}$$

(9) Note that

$$x \in f^{-1}\Big(\bigcap_{i\in I} B_i\Big) \iff f(x) \in \bigcap_{i\in I} B_i \iff f(x) \in B_i \text{ for each } i \in I$$

$$\iff x \in f^{-1}(B_i) \text{ for each } i \in I \iff x \in \bigcap_{i\in I} f^{-1}(B_i).$$

(10) Observe that

$$\begin{aligned} x \in f^{-1}(B^c) &\iff f(x) \in B^c \iff f(x) \notin B \\ &\iff x \notin f^{-1}(B) \iff x \in \big[f^{-1}(B)\big]^c. \end{aligned}$$

Problem 1.2. *For two sets A and B show that the following statements are equivalent:*

a. $A \subseteq B$;
b. $A \cup B = B$;
c. $A \cap B = A$.

Solution. (a) $\Longrightarrow$ (b) Clearly, $B \subseteq A \cup B$ holds. On the other hand, if $x \in A \cup B$, then $x \in A$ or $x \in B$, and so in either case, $x \in B$. This means $A \cup B \subseteq B$, and hence, $A \cup B = B$.
(b) $\Longrightarrow$ (c) By part (1) of the preceding problem, we have

$$A \cap B = A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cup (A \cap B) = A.$$

(c) $\Longrightarrow$ (a) Clearly, $A = A \cap B \subseteq B$.

Problem 1.3. *Show that $(A \Delta B)\Delta C = A\Delta(B \Delta C)$ holds for every triplet of sets A, B, and C.*

Solution. Note first that for any three sets X, Y, and Z we have

$$X \Delta Y \setminus Z = [X \setminus (Y \cup Z)] \cup [Y \setminus (X \cup Z)]$$

and

$$Z \setminus (X \Delta Y) = [Z \setminus (X \cup Y)] \cup [X \cap Y \cap Z].$$

For instance, to verify the first identity, note that

$$\begin{aligned} x \in X\Delta Y \setminus Z &\iff [x \in X \setminus Y \text{ or } x \in Y \setminus X] \text{ and } x \notin Z \\ &\iff [x \in X, x \notin Y, \text{ and } x \notin Z] \text{ or } [x \in Y,\ x \notin X, \text{ and } x \notin Z] \\ &\iff x \in [X \setminus (Y \cup Z)] \cup [Y \setminus (X \cup Z)]. \end{aligned}$$

Thus,

$$\begin{aligned}(A\Delta B)\Delta C &= [(A\Delta B)\setminus C]\cup[C\setminus(A\Delta B)]\\ &= [A\setminus(B\cup C)]\cup[B\setminus(A\cup C)]\cup[C\setminus(A\cup B)]\cup[A\cap B\cap C]\\ &= \{[A\setminus(B\cup C)]\cup(A\cap B\cap C)\}\cup\{[B\setminus(C\cup A)]\cup[C\setminus(B\cup A)]\}\\ &= [A\setminus(B\Delta C)]\cup[(B\Delta C)\setminus A]\\ &= A\Delta(B\Delta C).\end{aligned}$$

Problem 1.4. *Give an example of a function $f: X \to Y$ and two subsets A and B of X such that $f(A\cap B) \neq f(A)\cap f(B)$.*

Solution. Define $f:\{0,1\}\to\{0,1\}$ by $f(0) = f(1) = 0$. If $A = \{0\}$ and $B = \{1\}$, then $f(A\cap B) = \emptyset \neq \{0\} = f(A)\cap f(B)$.

Problem 1.5. *For a function $f: X \to Y$, show that the following three statements are equivalent:*

a. *f is one-to-one.*
b. *$f(A\cap B) = f(A)\cap f(B)$ holds for all $A,\ B\in \mathcal{P}(X)$.*
c. *For every pair of disjoint subsets A and B of X, we have $f(A)\cap f(B) = \emptyset$.*

Solution. $(a) \Longrightarrow (b)$ If $y\in f(A)\cap f(B)$, then there exist $a\in A$ and $b\in B$ with $y = f(a) = f(b)$. Since f is one-to-one, $a = b\in A\cap B$, and so $y\in f(A\cap B)$. Thus, $f(A)\cap f(B)\subseteq f(A\cap B)\subseteq f(A)\cap f(B)$.
$(b) \Longrightarrow (c)$ Obvious.
$(c) \Longrightarrow (a)$ Let $f(a) = f(b)$. If $a\neq b$, then the two sets $A = \{a\}$ and $B = \{b\}$ satisfy $A\cap B = \emptyset$, while $f(A)\cap f(B) = \{f(a)\}\neq\emptyset$.

Problem 1.6. *Let $f: X\to Y$ be a function. Show that $f(f^{-1}(A))\subseteq A$ for all $A\subseteq Y$, and $B\subseteq f^{-1}(f(B))$ for all $B\subseteq X$.*

Solution. Clearly, $x\in f^{-1}(A)$ if and only if $f(x)\in A$. Thus, $f(f^{-1}(A))\subseteq A$. Similarly, $x\in f^{-1}(f(B))$ if and only if $f(x)\in f(B)$, and so $B\subseteq f^{-1}(f(B))$ holds.

Problem 1.7. *Show that a function $f: X\to Y$ is onto if and only if $f(f^{-1}(B)) = B$ holds for all $B\subseteq Y$.*

Solution. Assume that f is onto and $B\subseteq Y$. If $b\in B$, then there exists some $a\in X$ with $f(a) = b$; clearly, $a\in f^{-1}(B)$. Thus, $b = f(a)\in f(f^{-1}(B))$, and so $B\subseteq f(f^{-1}(B))\subseteq B$ holds.

For the converse, note that the relation $f(f^{-1}(\{b\})) = \{b\}$ implies $f^{-1}(\{b\}) \neq \emptyset$ for each $b \in Y$ so that f is onto.

Problem 1.8. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If $A \subseteq Z$, show that*

$$(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)).$$

Solution. Note that

$$x \in (g \circ f)^{-1}(A) \iff g(f(x)) \in A \iff f(x) \in g^{-1}(A) \iff x \in f^{-1}(g^{-1}(A)).$$

Problem 1.9. *Show that the composition of functions satisfies the associative law. That is, show that if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} V$, then $(h \circ g) \circ f = h \circ (g \circ f)$.*

Solution. Observe that for each $x \in X$ we have

$$[(h \circ g) \circ f](x) = h \circ g(f(x)) = h(g(f(x))) = h((g \circ f)(x)) = [h \circ (g \circ f)](x).$$

Therefore, $(h \circ g) \circ f = h \circ (g \circ f)$.

Problem 1.10. *Let $f: X \to Y$. Show that the relation $\mathcal{R}$ on X, defined by $x_1 \mathcal{R} x_2$ whenever $f(x_1) = f(x_2)$, is an equivalence relation.*

Solution. We must show that the relation $\mathcal{R}$ is reflexive, symmetric, and transitive.

Reflexivity: Note that $f(x) = f(x)$ implies $x \mathcal{R} x$ for each $x \in X$.

Symmetry: Let $x_1 \mathcal{R} x_2$. Then, $f(x_1) = f(x_2)$ or $f(x_2) = f(x_1)$, so that $x_2 \mathcal{R} x_1$.

Transitivity: If $x_1 \mathcal{R} x_2$ and $x_2 \mathcal{R} x_3$, then $f(x_1) = f(x_2)$ and $f(x_2) = f(x_3)$ both hold. It follows that $f(x_1) = f(x_3)$, and so $x_1 \mathcal{R} x_3$.

Problem 1.11. *If X and Y are sets, then show that*

$$\mathcal{P}(X) \cap \mathcal{P}(Y) = \mathcal{P}(X \cap Y) \quad \textit{and} \quad \mathcal{P}(X) \cup \mathcal{P}(Y) \subseteq \mathcal{P}(X \cup Y).$$

Solution. (a) Note that

$$\begin{aligned} A \in \mathcal{P}(X) \cap \mathcal{P}(Y) &\iff A \subseteq X \text{ and } A \subseteq Y \\ &\iff A \subseteq X \cap Y \iff A \in \mathcal{P}(X \cap Y). \end{aligned}$$

(b) Clearly,

$$A \in \mathcal{P}(X) \cup \mathcal{P}(Y) \Longrightarrow A \subseteq X \text{ or } A \subseteq Y \Longrightarrow A \subseteq X \cup Y \Longrightarrow A \in \mathcal{P}(X \cup Y).$$

If X and Y are two nonempty disjoint sets, then $X \cup Y \notin \mathcal{P}(X) \cup \mathcal{P}(Y)$, and so equality is seldom valid.

2. COUNTABLE AND UNCOUNTABLE SETS

Problem 2.1. *Show that the set of all rational numbers is countable.*

Solution. Let $\mathcal{Q}$ be the set of rational numbers and let $\mathcal{Q}^+ = \{r \in \mathcal{Q}\colon\ r > 0\}$. Then the function $f\colon \mathbb{N} \times \mathbb{N} \to \mathcal{Q}^+$ defined by $f(m,n) = \frac{m}{n}$ is onto. The conclusion now follows from Theorems 2.7 and 2.5.

Problem 2.2. *Show that the set of all finite subsets of a countable set is countable.*

Solution. We can assume that $A = \{p_1, p_2, \dots\}$ is the set of all prime numbers. Let $\mathcal{F}$ denote the collection of all finite subsets of A. Define $f\colon \mathcal{F} \to \mathbb{N}$ by $f(F) =$ the product of the elements of F, for each $F \in \mathcal{F}$. Then f is one-to-one, and the conclusion follows from Theorem 2.5.

Problem 2.3. *Show that a union of an at-most countable collection of sets, each of which is finite, is an at-most countable set.*

Solution. This follows immediately from Theorem 2.6.

Problem 2.4. *Let A be an uncountable set and let B be a countable subset of A. Show that A is equivalent to $A \setminus B$.*

Solution. Let $B = \{b_1, b_2, \dots\}$. Since A is uncountable, the set $A \setminus B$ is also uncountable. Let $C = \{c_1, c_2, \dots\}$ be a countable subset of $A \setminus B$. Now define $f\colon A \setminus B \to A$ by

$$f(x) = \begin{cases} x, & \text{if } x \notin C; \\ c_{n+1}, & \text{if } x = c_{2n+1}\ (n = 0, 1, 2, \dots); \\ b_n, & \text{if } x = c_{2n}\ (n = 1, 2, \dots). \end{cases}$$

Then f is one-to-one and onto, proving that $A \approx A \setminus B$.

Problem 2.5. *Assume that $f\colon A \to B$ is a surjective (onto) function between two sets. Establish the following:*

a. card $B \leq$ card A.
b. *If A is countable, then B is at-most countable.*

Solution. (a) Consider the family $\{f^{-1}(b)\colon b \in B\}$. Clearly, this is a family of disjoint subsets of A. By the Axiom of Choice there exists a subset C of A such that $C \cap f^{-1}(b)$ consists precisely of one element of A for each $b \in B$. The conclusion now follows by observing that $f\colon C \to B$ is one-to-one and onto.

(b) This follows immediately from part (a).

Problem 2.6. *Show that two nonempty sets A and B are equivalent if and only if there exists a function from A onto B and a function from B onto A.*

Solution. If A and B are equivalent, then there exists a function $f\colon A \to B$ which is one-to-one and onto. Clearly, $f^{-1}\colon B \to A$ is a surjective function.

For the converse, assume that there exists a function from A onto B and a function from B onto A. A glance at the preceding problem guarantees that $\operatorname{card} B \leq \operatorname{card} A$ and $\operatorname{card} A \leq \operatorname{card} B$. Now, use the Schröder–Bernstein theorem to conclude that A and B are equivalent sets.

Problem 2.7. *Show that if a finite set X has n elements, then its power set $\mathcal{P}(X)$ has 2^n elements.*

Solution. We shall use induction on n. Assume that $\{1, \ldots, n\}$ has 2^n subsets. Then the subsets of the set $\{1, \ldots, n, n+1\}$ consist of:

a. The subsets of $\{1, \ldots, n\}$, which are 2^n altogether; and
b. The subsets of the form $A \cup \{n+1\}$, where A is a subset of $\{1, \ldots, n\}$, again 2^n altogether.

Thus, the number of subsets of $\{1, \ldots, n, n+1\}$ is $2^n + 2^n = 2^{n+1}$.

A direct proof goes as follows. Notice that the number of subsets of $\{1, 2, \ldots, n\}$ having k elements (where $0 \leq k \leq n$) is precisely $\binom{n}{k}$. So, the total number of subsets of $\{1, 2, \ldots, n\}$ is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n,$$

where the last equality holds true by virtue of the binomial theorem.

Problem 2.8. *Show that the set of all sequences with values 0 or 1 is uncountable.*

Solution. For each subset A of $\mathbb{N}$ define the sequence $f(A) = \{x_n\}$ by $x_n = 1$ if $n \in A$ and $x_n = 0$ if $n \notin A$. Then f defines a function from $\mathcal{P}(\mathbb{N})$ onto

the sequences with values 0 and 1. Since f is clearly one-to-one and onto, the conclusion follows from Theorem 2.8.

Problem 2.9. *If $2 = \{0, 1\}$, then show that $2^X \approx \mathcal{P}(X)$ for every set X.*

Solution. Define $f: \mathcal{P}(X) \to 2^X$ by $A \longmapsto f_A$, where $f_A(x) = 1$ if $x \in A$ and $f_A(x) = 0$ if $x \notin A$. Note that f is one-to-one and onto. Therefore, $2^X \approx \mathcal{P}(X)$.

Problem 2.10. *Any complex number that is a root of a (nonzero) polynomial with integer coefficients is called an* **algebraic number**. *Show that the set of all algebraic numbers is countable.*

Solution. Let $\mathcal{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$. Fix $n \geq 1$. Since every polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is determined uniquely by $(a_0, a_1, \dots, a_n)$, it is easy to see that the nonzero polynomials of degree $\leq n$ with integer coefficients are in one-to-one correspondence with the countable set $\mathcal{Z}^{n+1} \setminus \{(0, 0, \dots, 0)\}$. Let $\{p_1, p_2, \dots\}$ be an enumeration of all these polynomials. By the Fundamental Theorem of Algebra, the set $A_k = \{x \in \mathbf{C}\colon\ p_k(x) = 0\}$ is a finite set. Thus, the set of all zeros of the polynomials $\{p_1, p_2, \dots\}$ of degree $\leq n$ is precisely the set $R_n = \bigcup_{k=1}^{\infty} A_k$, which (by Theorem 2.6) is a countable set. Now, note that the set of all algebraic numbers is $\bigcup_{n=1}^{\infty} R_n$, which—as a countable union of countable sets—is itself countable; see Theorem 2.6.

Problem 2.11. *For an arbitrary function $f: \mathbf{R} \to \mathbf{R}$ show that the set*

$$A = \Big\{a \in \mathbf{R}\colon\ \lim_{x \to a} f(x) \text{ exists and } \lim_{x \to a} f(x) \neq f(a)\Big\}$$

is at-most countable.

Solution. Let $\mathcal{I}$ denote the set of all open subintervals of $\mathbf{R}$ with rational endpoints and note that $\mathcal{I}$ is a countable set. Also, let $\mathcal{Q}$ denote the countable set of all rational numbers of $\mathbf{R}$.

For each rational real number r, let

$$A_r = \Big\{a \in A\colon\ \text{Either } f(a) < r < \lim_{x \to a} f(x) \text{ or } \lim_{x \to a} f(x) < r < f(a)\Big\}.$$

Clearly, $A = \bigcup_{r \in \mathcal{Q}} A_r$ holds. Thus, in order to establish that A is at most countable, it suffices to show that each A_r is at-most countable.

So, fix some $r \in \mathcal{Q}$ and $a \in A_r$ and assume (without loss of generality) that $f(a) < r < \lim_{x \to a} f(x)$. Then there exists as $\delta > 0$ such that $a - \delta < y < a + \delta$ and $y \neq a$ imply $f(y) > r$. Next, pick an open interval I_a with rational endpoints (i.e., $I_a \in \mathcal{I}$) such that $a \in I_a$ and $I_a \subseteq (a - \delta, a + \delta)$. Since $f(y) > r$

holds for each $y \in I_a$ with $y \neq a$, we see that $y \notin A_r$ for each $y \in I_a \setminus \{a\}$. In particular, note that $A_r \cap I_a = \{a\}$.

Thus, we have established a mapping $a \longmapsto I_a$ from A_r into $\mathcal{I}$ (which in view of $A_r \cap I_a = \{a\}$ for each $a \in A_r$) is also one-to-one. This implies that A_r is at-most countable, and hence, A is likewise at-most countable.

Problem 2.12. *Show that the set of real numbers is uncountable by proving the following:*

a) $(0, 1) \approx \mathbb{R}$; *and*

b) $(0, 1)$ *is uncountable.*

Solution. (a) The function $f\colon (0, 1) \to \mathbb{R}$ defined by the formula $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$ is one-to-one and onto.

(b) If $(0, 1)$ is countable, then let $\{x_1, x_2, \ldots\}$ be one enumeration of $(0, 1)$. For each n write $x_n = 0.d_{n1}d_{n2}\cdots$ in its decimal expansion, where each d_{ij} is $0, 1, \ldots,$ or 9. Now, consider the real number y of $(0, 1)$ whose decimal expansion $y = 0.y_1y_2\cdots$ satisfies $y_n = 1$ if $d_{nn} \neq 1$ and $y_n = 2$ if $d_{nn} = 1$. An easy argument now shows (how?) that $y \neq x_n$ for each n, which is a contradiction. Hence, the interval $(0, 1)$ is an uncountable set.

Problem 2.13. *Using mathematical induction prove the following:*

a. *If* $a \geq -1$, *then* $(1+a)^n \geq 1+na$ *for* $n = 1, 2, \ldots$ *(Bernoulli's inequality).*

b. *If* $0 < a < 1$, *then* $1 + 3^n a > (1 + a)^n$ *for* $n = 1, 2, \ldots$.

c. $\cos(n\pi) = (-1)^n$ *for* $n = 1, 2, \ldots$.

Solution. (a) Let $a \geq -1$. For $n = 1$ the inequality is trivially true; in fact, it is an equality. For the induction step, assume that $(1 + a)^n \geq 1 + na$ holds true for some n. Since $1 + a > 0$ is assumed to be true, it follows that

$$\begin{aligned}(1 + a)^{n+1} &= (1 + a)(1 + a)^n \geq (1 + a)(1 + na) = 1 + na + a + na^2\\ &= 1 + (n + 1)a + na^2 \geq 1 + (n + 1)a,\end{aligned}$$

which is the desired inequality when n takes the value $n + 1$. This completes the induction.

(b) Assume $0 < a < 1$. Since $1 + 3a > 1 + a$, the desired inequality is true for $n = 1$. For the inductive step assume $1 + 3^n a > (1 + a)^n$. Then, taking into account that $0 < a < 1$, we see that

$$\begin{aligned}(1 + a)^{n+1} &= (1 + a)(1 + a)^n < (1 + a)(1 + 3^n a)\\ &= 1 + 3^n a + a + 3^n a^2 = 1 + (3^n + 3^n a + 1)a\\ &< 1 + (3^n + 3^n + 3^n)a = 1 + 3 \cdot 3^n a = 1 + 3^{n+1} a,\end{aligned}$$

which is the desired inequality valid when n is replaced by $n+1$. By the Principle of Mathematical Induction, the inequality is true for every natural number n.

(c) For $n = 1$, we have $\cos(1 \cdot \pi) = \cos \pi = -1 = (-1)^1$. Now, assume that $\cos(n\pi) = (-1)^n$. Then, using the trigonometric formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$, we see that

$$\begin{aligned}\cos\big[(n+1)\pi\big] &= \cos(n\pi + \pi) = \cos(n\pi)\cos\pi - \sin(n\pi)\sin\pi \\ &= (-1)^n(-1) - \sin(n\pi)\cdot 0 = (-1)^{n+1},\end{aligned}$$

and the induction is complete.

Problem 2.14. *Show that the Well-Ordering Principle implies the Principle of Mathematical Induction.*

Solution. Let $S \subseteq \mathbb{N}$ satisfy

a. $1 \in S$, and
b. $n+1 \in S$ whenever $n \in S$.

We must show that $S = \mathbb{N}$, or equivalently that $\mathbb{N} \setminus S = \emptyset$.

To this end, assume by way of contradiction that we have $\mathbb{N} \setminus S \neq \emptyset$. Then, by the Well Ordering Principle, $n = \min(\mathbb{N} \setminus S)$ exists. Clearly, $1 < n \in \mathbb{N} \setminus S$. Thus, $n - 1 \in S$, and consequently $n = (n-1)+1 \in S$, a contradiction. Therefore, $\mathbb{N} \setminus S = \emptyset$ or $S = \mathbb{N}$.

Problem 2.15. *Show that the Principle of Mathematical Induction implies the Well-Ordering Principle.*

Solution. Assume that the Principle of Mathematical Induction is true. Consider the subset S of $\mathbb{N}$ consisting of all natural numbers n with the property: whenever a nonempty subset A of $\mathbb{N}$ contains a natural number $m \leq n$, then A has a least element. To establish the Well-Ordering Principle, we need to show that $S = \mathbb{N}$.

To this end, note that $1 \in S$. Now assume that $n \in S$. Also, assume that a nonempty subset A of $\mathbb{N}$ contains some natural number $m \leq n+1$. If A contains a natural number $k < n+1$, then A also contains a natural number (namely k itself) less than or equal to n, and so, in view of $n \in S$, A must have a least element. On the other hand, if A does not contain any natural number strictly less that $n+1$, it follows that $n+1 \in A$, in which case $n+1$ is the least element of A. Therefore, $n+1 \in S$, and so by the validity of the Principle of Mathematical Induction, we infer that $S = \mathbb{N}$.

3. THE REAL NUMBERS

Problem 3.1. *If* $a \vee b = \max\{a, b\}$ *and* $a \wedge b = \min\{a, b\}$, *then show that*

$$a \vee b = \tfrac{1}{2}(a + b + |a - b|) \quad \textit{and} \quad a \wedge b = \tfrac{1}{2}(a + b - |a - b|).$$

Solution. Since all expressions do not change their values if we interchange a and b, we can assume $a \geq b$. Thus,

$$\tfrac{1}{2}(a + b + |a - b|) = \tfrac{1}{2}(a + b + a - b) = a = a \vee b,$$

and

$$\tfrac{1}{2}(a + b - |a - b|) = \tfrac{1}{2}[a + b - (a - b)] = b = a \wedge b.$$

Problem 3.2. *Show that* $\big||a| - |b|\big| \leq |a + b| \leq |a| + |b|$ *for all* $a, b \in \mathbf{R}$.

Solution. From $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, it follows that

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

So, $|a + b| \leq |a| + |b|$.

Substituting $a - b$ in the place of a, we get $|a| \leq |a - b| + |b|$ so that $|a| - |b| \leq |a - b|$. Interchanging a and b yields $-(|a| - |b|) \leq |a - b|$, and so $\big||a| - |b|\big| \leq |a - b|$ also holds.

Problem 3.3. *Show that the real numbers* $\sqrt{2}$ *and* $\sqrt{2} + \sqrt{3}$ *are irrational numbers.*

Solution. Assume by way of contradiction that $\sqrt{2} = \frac{m}{n}$ with $m, n \in \mathbf{N}$. We can suppose that m and n have no common positive divisors other than 1. Squaring, we get $m^2 = 2n^2$. This implies that m is even, i.e., $m = 2k$ for some $k \in \mathbf{N}$ (otherwise $m = 2k + 1$ implies that m^2 is odd, a contradiction). It follows that $4k^2 = 2n^2$, or $n^2 = 2k^2$, which in turn implies that n is even, i.e., $n = 2\ell$ for some $\ell \in \mathbf{N}$. But then, m and n have the common factor 2, which is a contradiction. Hence, $\sqrt{2}$ is not a rational number. (This simple proof is due to Eudoxus.)

With a different and more elegant proof one can establish the following general result:

- *The square root* $\sqrt{k}$ *of a natural number* k *is a rational number if and only if* k *is a complete square, i.e.,* $k = p^2$ *for some* $p \in \mathbf{N}$.

If $k = p^2$, then clearly $\sqrt{k} = p \in \mathbb{N}$. On the other hand, if $\sqrt{k}$ is a rational number, then $\sqrt{k}$ is a rational root of the polynomial $p(x) = x^2 - k$. But the positive rational roots of this polynomial are of the form $\frac{m}{n}$, where $m \in \mathbb{N}$ is a divisor of k and $n \in \mathbb{N}$ is a divisor of 1. Thus, $\sqrt{k} = m \in \mathbb{N}$, and so $k = m^2$.

To see that $\sqrt{2}+\sqrt{3}$ is not a rational number, assume by way of contradiction that $\sqrt{2}+\sqrt{3} = r > 0$ is a rational number. Then $\sqrt{3} = r - \sqrt{2}$ and by squaring, we get $3 = r^2 - 2r\sqrt{2} + 2$. This implies $\sqrt{2} = \frac{r^2-1}{2r}$, a rational number, contrary to our previous conclusion. Hence, $\sqrt{2}+\sqrt{3}$ is an irrational number.

Problem 3.4. *Show that between any two distinct real numbers there is an irrational number.*

Solution. Let $a < b$. Choose a rational number r with $a < r < b$, and then select some n so that $0 < \frac{\sqrt{2}}{n} < b-r$. Note that the irrational number $x = r + \frac{\sqrt{2}}{n}$ satisfies $a < x < b$.

Alternatively: Note that the open interval (a, b) is uncountable, while the set of all rational numbers is countable.

Problem 3.5. *This problem will introduce (by steps) the familiar process of subtraction in the framework of the axiomatic foundation of real numbers.*

a. *Show that the element 0 is uniquely determined, i.e., show that if $x + 0^* = x$ for all $x \in \mathbf{R}$ and some $0^* \in \mathbf{R}$, then $0^* = 0$.*
b. *Show that the* **cancellation law of addition** *is valid, i.e., show that $x + a = x + b$ implies $a = b$.*
c. *Use the cancellation law of addition to show that $0 \cdot a = 0$ for all $a \in \mathbf{R}$.*
d. *Show that for each real number a the real number $-a$ is the unique real number that satisfies the equation $a + x = 0$. (The real number $-a$ is called the* **negative** *of a.)*
e. *Show that for any two given real numbers a and b, the equation $a + x = b$ has a unique solution, namely $x = b + (-a)$. The* **subtraction** *operation $-$ of $\mathbf{R}$ is now defined by $a - b = a + (-b)$; the real number $a - b$ is also called the* **difference** *of b from a.*
f. *For any real numbers a and b show that $-(-a) = a$ and $-(a+b) = -a-b$.*

Solution. (a) Assume than another element $0^* \in \mathbf{R}$ satisfies $0^* + x = x + 0^* = x$ for all $x \in \mathbf{R}$. Letting $x = 0$, we get $0^* + 0 = 0$. Now, recalling that $0 + y = y + 0 = y$ also holds for all $y \in \mathbf{R}$, letting $y = 0^*$ yields $0^* = 0^* + 0 = 0$.
(b) Let $x + a = x + b$. By Axiom 5 there exists some $z \in \mathbf{R}$ such that $z + x = x + z = 0$. So,

$$a = 0 + a = (z + x) + a = z + (x + a) = z + (x + b) = (z + x) + b = 0 + b = b.$$

(c) Clearly,

$$0 \cdot a + 0 = 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a,$$

and so by the cancellation law of addition, $0 \cdot a = 0$ for each $a \in \mathbf{R}$.
(d) Assume $a + x = 0$. Since $a + (-a) = 0$, we see that $a + x = a + (-a)$, and so, by the cancellation law we have established in (b) above, $x = -a$; the negative of a.
(e) If $a + z = a + y = b$, then by the cancellation law, we get $z = y$. Thus, given a and b, the equation $a + x = b$ has at-most one solution $x \in \mathbf{R}$. Since

$$a+\big[b+(-a)\big] = (a+b)+(-a) = (-a)+(a+b) = \big[(-a)+a\big]+b = 0+b = b,$$

we see that the only solution of the equation $a + x = b$ is $x = b + (-a)$. We denote this number by $b - a$ and call it the subtraction of a from b.
(f) A close look at the equation $a+(-a) = (-a)+a = 0$ guarantees immediately that $-(-a) = a$. Moreover, from

$$a+b+(-a-b) = a+b+\big[-a+(-b)\big] = \big[(a+b)+(-a)\big]+(-b) = b+(-b) = 0,$$

we easily infer that $-(a + b) = -a - b$.

Problem 3.6. *This problem introduces (by steps) the familiar process of division in the framework of the axiomatic foundation of real numbers.*

a. *Show that the element* 1 *is uniquely determined, i.e., show that if* $1^* \cdot x = x$ *for all* $x \in \mathbf{R}$ *and some* $1^* \in \mathbf{R}$, *then* $1^* = 1$.
b. *Show that the* **cancellation law of multiplication** *is valid, i.e., show that* $x \cdot a = x \cdot b$ *with* $x \neq 0$ *implies* $a = b$.
c. *Show that for each real number* $a \neq 0$ *the real number* a^{-1} *is the unique real number that satisfies the equation* $x \cdot a = 1$. *The real number* $x = a^{-1}$ *is called the* **inverse** (*or the* **reciprocal**) *of* a.
d. *Show that for any two given real numbers* a *and* b *with* $a \neq 0$, *the equation* $ax = b$ *has a unique solution, namely* $x = a^{-1}b$. *The* **division** *operation* $\div$ (*or* $/$) *of* $\mathbf{R}$ *is now defined by* $b \div a = a^{-1}b$; *as usual, the real number* $b \div a$ *is also denoted by* b/a *or* $\frac{b}{a}$.
e. *For any two nonzero* $a, b \in \mathbf{R}$ *show that* $(a^{-1})^{-1} = a$ *and* $(ab)^{-1} = a^{-1}b^{-1}$.
f. *Show that* $\frac{a}{1} = a$ *for each* a, $\frac{0}{b} = 0$ *for each* $b \neq 0$, *and* $\frac{a}{a} = 1$ *for each* $a \neq 0$.

Solution. (a) Assume that some real number 1^* satisfies $1^* \cdot x = x \cdot 1^* = x$ for each $x \in \mathbf{R}$. In particular, letting $x = 1$, we get $1^* \cdot 1 = 1$. Since $y \cdot 1 = y$ for all $y \in \mathbf{R}$, letting $y = 1^*$ yields $1^* = 1^* \cdot 1 = 1$. So, 1 is the only real number r which satisfies $r \cdot x = x$ for each $x \in \mathbf{R}$.

(b) Assume $x \cdot a = x \cdot b$ with $x \neq 0$. By Axiom 7 there exists a real number $y \in \mathbf{R}$ such that $y \cdot x = 1$. Now, observe that

$$a = 1 \cdot a = (yx)a = y(xa) = y(xb) = (yx)b = 1 \cdot b = b.$$

(c) If $ax = ay = 1$ with $a \neq 0$, then by (b), we must have $x = y$. This shows that the reciprocal a^{-1} of a is uniquely determined.

(d) To see that the equation $ax = b$ with $a \neq 0$ has at-most one solution x, notice that if $ax = ay = b$, then by the cancellation law of multiplication, we have $x = y$. Moreover, notice that

$$a \cdot (a^{-1}b) = (a \cdot a^{-1})b = 1 \cdot b = b.$$

The above show that the equation $ax = b$ with $a \neq 0$ has the unique solution $x = a^{-1}b$.

(e) If $a \neq 0$, then the equation $a \cdot a^{-1} = 1$ readily says that $(a^{-1})^{-1} = a$. In addition, from

$$(ab) \cdot (b^{-1}a^{-1}) = a(b \cdot b^{-1})a^{-1} = a \cdot 1 \cdot a^{-1} = 1,$$

we easily obtain $(ab)^{-1} = b^{-1}a^{-1}$.

(f) Since $1 \cdot a = a$, we obtain $\frac{a}{1} = a$ for each $a \in \mathbf{R}$. The equation $b \cdot 0 = 0$ also implies that $\frac{0}{b} = 0$ for each $b \neq 0$. From $a \cdot 1 = a$, we get immediately $\frac{a}{a} = 1$ for all $a \neq 0$.

Problem 3.7. *Establish the following familiar properties of real numbers using the axioms of the real numbers together with the properties established in the previous two problems.*

i. **The zero product rule:** $ab = 0$ *if and only if either* $a = 0$ *or* $b = 0$.
ii. **The multiplication rule of signs:** $(-a)b = a(-b) = -(ab)$ *and* $(-a)(-b) = ab$ *for all* $a, b \in \mathbf{R}$.
iii. **The multiplication rule for fractions:** *For* $b, d \neq 0$ *and arbitrary real numbers* a, c *we have*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

In particular, if $\frac{a}{b} \neq 0$, *then* $(\frac{a}{b})^{-1} = \frac{b}{a}$.
iv. **The cancellation law of division:** *If* $a \neq 0$ *and* $x \neq 0$, *then* $\frac{bx}{ax} = \frac{b}{a}$ *for each real number* b.
v. **The division rule for fractions:** *Division by a fraction is the same as*

multiplication by the reciprocal of the fraction, i.e., whenever the fraction $\frac{a}{b} \div \frac{c}{d}$ is defined, we have

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Solution. (i) We already know from the previous problem that $0 \cdot b = 0$ for each $b \in \mathbf{R}$. On the other hand, if $ab = 0 (= a \cdot 0)$ and $a \neq 0$, the cancellation law of multiplication shows that $b = 0$.

(ii) Clearly,

$$ab+(-a)b = \left[a+(-a)\right]b=0 \cdot b=0 \quad \text{and} \quad ab+a(-b) = a\left[b+(-b)\right]=a \cdot 0=0,$$

and so $-(ab) = (-a)b = a(-b)$. This implies

$$(-a)(-b) = -[a(-b)] = -[-(ab)] = ab.$$

(iii) If $b, d \neq 0$, then

$$bd\left(\frac{a}{b} \cdot \frac{c}{d}\right) = \left(b \cdot \frac{a}{b}\right) \cdot \left(d \cdot \frac{c}{d}\right) = ac,$$

and this shows that $\frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d}$. Since $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = 1$, we see that $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

(iv) If $c = \frac{b}{a}$, then $ac = b$ and so $(ax)c = bx$ for each $x \neq 0$, which shows that $c = \frac{b}{a} = \frac{bx}{ax}$.

(v) Notice that the identity

$$\frac{c}{d} \cdot \frac{ad}{bc} = \frac{adc}{dbc} = \frac{a}{b}$$

guarantees $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \cdot \frac{d}{c}$.

Problem 3.8. *This problem establishes that there exists essentially one set of real numbers that satisfies the eleven axioms stated in Section 3. To see this, let* $\mathbf{R}$ *be a set of real numbers (i.e., a collection of objects that satisfies all eleven axioms stated in Section 3 of the text).*

a. *Show that* $1 > 0$.
b. *A real number a satisfies* $a = -a$ *if and only if* $a = 0$.
c. *If* $n = 1 + 1 + \cdots + 1$ *(where the sum has "n summands" all equal to 1), then show that these elements are all distinct; as usual, we shall call the collection* $\mathbf{N}$ *of all these numbers the natural numbers of* $\mathbf{R}$.
d. *Let Z consist of* $\mathbf{N}$ *together with its negative elements and zero; we shall call Z, of course, the set of integers of* $\mathbf{R}$. *Show that Z consists of distinct elements and that it is closed under addition and multiplication.*

e. *Define the set Q of rational numbers by $Q = \{\frac{m}{n}\colon\ m, n \in Z \text{ and } n \neq 0\}$. Show that Q satisfies itself axioms 1 through 10 and that*

$$a = \sup\{r \in Q\colon\ r \leq a\} = \inf\{s \in Q\colon\ a \leq s\}$$

holds for each $a \in \mathbf{R}$.

f. *Now, let $\mathbf{R}'$ be another set of real numbers and let Q' denote its rational numbers. If $1'$ denotes the unit element of $\mathbf{R}'$, then we write*

$$n' = 1' + 1' + \cdots + 1'$$

for the sum having "n-summands" all equal to $1'$. Now, define the function $f\colon Q \to Q'$ by

$$f(\tfrac{m}{n}) = \tfrac{m'}{n'}$$

and extend it to all of $\mathbf{R}$ via the formula

$$f(a) = \sup\{f(r)\colon\ r \leq a\}.$$

Show that $\mathbf{R}$ and $\mathbf{R}'$ essentially coincide by establishing the following:

i. *$a \leq b$ holds in $\mathbf{R}$ if and only if $f(a) \leq f(b)$ holds in $\mathbf{R}'$.*
ii. *f is one-to-one and onto.*
iii. *$f(a+b) = f(a)+f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in \mathbf{R}$.*

Solution. (a) Since $1 \neq 0$, we have two possibilities: either $1 > 0$ or $0 > 1$. If $0 > 1$, then (by Axiom 9) we have $0 + (-1) > 1 + (-1) = 0$ or $-1 > 0$, i.e., -1 is a positive number. Now, using Axiom 10, we infer that $0 \cdot (-1) \geq 1 \cdot (-1)$, or $0 \geq -1$, contrary to $-1 > 0$. Hence, $1 > 0$.

(b) Since $0 + 0 = 0$, we know that $-0 = 0$. Conversely, assume that a real number a satisfies $a = -a$. This implies $a + a = (1 + 1)a = 0$. However, since $1 > 0$, we have $1 + 1 \geq 1 + 0 = 1 > 0$, and so $1 + 1 \neq 0$. Consequently, from the zero product rule, $(1 + 1)a = 0$ implies $a = 0$.

(c) As shown in part (b) above, $1 + 1 \neq 0$ and in fact $1 + 1 \neq 1$; otherwise $1 + 1 = 1 = 1 + 0$ implies (in view of the cancellation law) $1 = 0$, which is impossible. Now, by induction, assume that

$$0 < 1 < 1 + 1 < 1 + 1 + 1 < \cdots < \underbrace{1 + 1 + \cdots + 1}_{n\text{-summands}}.$$

We claim that the real number $n + 1 = 1 + 1 + \cdots + 1 + 1$ (a sum of $n + 1$ summands) satisfies $n + 1 > n = 1 + 1 + \cdots + 1$ (where the last sum has n summands). Indeed, if $n + 1 \leq n$, then $(n + 1) + (-n) \leq n + (-n)$ or $1 \leq 0$, which is a contradiction. Hence, $n + 1 > n$ and the induction is complete.

(d) By part (c) we know that the natural numbers together with zero are all distinct real numbers. If $-m = -n$ with $m, n \in \mathbf{N}$, then $m = n$, which shows that distinct natural numbers have distinct negatives. If $m = -n$ with $m, n \in \mathbf{N}$, then

$m + n = 0$ contradicting (c), and so no natural number can be equal to a negative integer. It now follows that Z consists of distinct elements.

(e) Observe that if $\frac{m}{n}$ and $\frac{p}{q}$ are two rational numbers, then

$$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} \quad \text{and} \quad \frac{m}{n} \cdot \frac{p}{q} = \frac{mp}{nq},$$

and if $\frac{m}{n} \neq 0$, then $\left(\frac{m}{n}\right)^{-1} = \frac{n}{m}$. That is, Q is closed under addition, multiplication and inverses. Since all real numbers satisfy axioms 1 through 10, it follows that Q itself satisfies axioms 1 through 10 in its own right.

For the second part, fix $a \in \mathbf{R}$ and let $A = \{r \in Q\colon r \leq a\}$. Since there exists a rational number between $a - 1$ and a (see Theorem 3.4), A is nonempty, and clearly A is bounded from above by a. By the Completeness Axiom (Axiom 11), $\sup A$ exists in $\mathbf{R}$ and satisfies $\sup A \leq a$.

Now, let $\epsilon > 0$. By Theorem 3.4, there exists some rational number r such that $a - \epsilon < r < a$. Clearly, $r \in A$, and so $a - \epsilon < \sup A$, or $a < \sup A + \epsilon$, holds for all $\epsilon > 0$. This implies $a \leq \sup A$, and hence $a = \sup A$. The equality, $a = \inf\{s \in Q\colon a \leq s\}$ can be proven in a similar manner.

(f) Notice that the mapping is well defined. That is, if $\frac{m}{n} = \frac{p}{q}$ in Q, then $f\left(\frac{m}{n}\right) = f\left(\frac{p}{q}\right)$. Indeed, since $\frac{m}{n} = \frac{p}{q}$ is equivalent to $mq = np$, we see that $m'q' = n'q'$ or $\frac{m'}{n'} = \frac{p'}{q'}$. Now, let us verify properties (i), (ii), and (iii).

i. $a \leq b$ holds in $\mathbf{R}$ if and only if $f(a) \leq f(b)$ holds in $\mathbf{R}'$.

Note first that two rational numbers $r, s \in Q$ satisfy $r \leq s$ if and only if $r' \leq s'$. Indeed, to see this it suffices to assume that r and s are positive rational numbers (why?). We have

$$r = \frac{m}{n} \leq s = \frac{p}{q} \iff mq \leq np \iff m'q' \leq n'q' \iff r' = \frac{m'}{n'} \leq \frac{p'}{q'} = s'.$$

Now, let $a \leq b$. Then $\{r \in Q\colon r \leq a\} \subseteq \{s \in Q\colon s \leq b\}$, and from this it easily follows that $f(a) \leq f(b)$. For the converse, assume that $f(a) \leq f(b)$. If $a \leq b$ is not true, then we must have $b < a$. But then, by Theorem 3.4 there exist two rational numbers $r, s \in Q$ such that $b < r < s < a$. This implies $f(b) \leq r' < s' \leq f(a)$, a contradiction.

ii. f is one-to-one and onto.

To see that f is onto, let $a' \in \mathbf{R}'$. Then by Theorem 3.4,

$$a' = \sup\{t \in Q'\colon t \leq a'\}.$$

If, we let $S = \{r \in Q\colon r' \leq a'\}$, then this set is bounded above in $\mathbf{R}$ (why?) and so $a = \sup S$ exists in $\mathbf{R}$. Moreover, notice that

$$\{t \in Q'\colon t \leq a'\} = \{r'\colon r \in Q \text{ and } r \leq a\}.$$

Now, it is easy to see that $f(a) = a'$.

To verify that f is one-to-one assume $f(a) = f(b)$. Then by part (i) we have $a \le b$ and $b \le a$, i.e., $a = b$.

iii. $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in \mathbf{R}$.

We verify the additivity property only and leave the multiplicative property for the reader. Clearly, $f(r+s) = f(r) + f(s)$ holds for all rational numbers r, s. Now, fix $a, b \in \mathbf{R}$ and assume $r, s \in \mathcal{Q}$ satisfy $r \le a$ and $s \le b$. Then $f(r) = r' \le f(a)$ and $f(s) = s' \le f(b)$. Since $r + s \in \mathcal{Q}$ and $r + s \le a + b$, we see that $f(r) + f(s) = r' + s' = f(r+s) \le f(a+b)$. This easily implies

$$f(a) + f(b) \le f(a+b).$$

For the reverse inequality, let $\epsilon' > 0$ in $\mathbf{R}'$. Then there exist rational numbers $r, s \in \mathcal{Q}$ with $r \le a$ and $s \le b$ such that $f(a) - \epsilon' < f(r)$ and $f(b) - \epsilon' < f(s)$. Since $r + s \le a + b$, it follows that $f(a) + f(b) - 2\epsilon' < f(r) + f(s) = f(r+s) \le f(a+b)$ for each $\epsilon' > 0$. This guarantees $f(a) + f(b) \le f(a+b)$, and therefore $f(a+b) = f(a) + f(b)$.

Problem 3.9. *Consider a two-point set $R = \{0, 1\}$ equipped with the following operations:*

a. *Addition* $(+): 0 + 0 = 0, 0 + 1 = 1 + 0 = 1$ *and* $1 + 1 = 0$,
b. *Multiplication* $(\cdot): 0 \cdot 1 = 1 \cdot 0 = 0$ *and* $1 \cdot 1 = 1$, *and*
c. *Ordering*: $0 \ge 0$, $1 \ge 1$ *and* $1 \ge 0$.

Does R with the above operations satisfy all eleven axioms defining the real numbers? Explain your answer.

Solution. It satisfies all axioms except Axiom 9, which states that:

- *If $x \ge y$ and $z \ge 0$, then $x + z \ge y + z$.*

To see this, assume that Axiom 9 is valid. We distinguish two cases.

CASE I: $1 > 0$.

In this case, we must have $0 = 1 + 1 \ge 0 + 1 = 1$, which contradicts $1 > 0$.

CASE II: $0 > 1$.

This implies $1 = 0 + 1 \ge 1 + 1 = 0$, which again contradicts $0 > 1$. Thus, Axiom 9 does not hold in this case.

It should be noticed that Axiom 9 is the one that guarantees that $1 + 1$ (i.e, the number 2) is distinct from 0 and 1; and, of course, it is the axiom that establishes (as we saw in part (b) of Problem 3.8) the existence of the set of integers.

Problem 3.10. *Consider the set of rational numbers $\mathcal{Q}$ equipped with the usual operations of addition, multiplication, and ordering. Why doesn't $\mathcal{Q}$ coincide with the set of real numbers?*

Solution. The set of rational numbers satisfies all the axioms of real numbers except the completeness axiom. This was proven in part (e) of Problem 3.8. To see that Q does not satisfy the completeness axiom, assume by way of contradiction that it does. Consider the set

$$S = \{0 \le r \in Q\colon r^2 \le 2\}.$$

Then S is nonempty and bounded from above in Q (why?), and so $b = \sup S$ exists in Q. Now, repeat the proof of Theorem 3.5 to conclude that $b^2 = 2$, i.e., that $b = \sqrt{2}$. However, we proved in Problem 3.3 that $\sqrt{2}$ is not a rational number, and we have reached a contradiction. Hence, Q does not satisfy the completeness axiom and it cannot coincide with the set of real numbers.

Problem 3.11. *This problem establishes the familiar rules of "exponents" based on the axiomatic foundation of real numbers. To avoid unnecessary notation, we shall assume that all real numbers encountered here are positive—and so by Theorem 3.5, all non-negative real numbers have unique roots. As usual, the "integer" powers are defined by*

$$a^n = \underbrace{a \cdot a \cdots a}_{n\text{-factors}}, \quad a^0 = 1, \quad a^1 = a, \quad \textit{and} \quad a^{-n} = \frac{1}{a^n}.$$

Extending this to rational numbers, for each $m, n \in \mathbb{N}$ we define

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \textit{and} \quad a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}}.$$

Establish the following properties:

a. $a^{\frac{m}{n}} = (\sqrt[n]{a}\,)^m$ *for all* $m, n \in \mathbb{N}$.
b. *If* $m, n, p, q \in \mathbb{N}$ *satisfy* $\frac{m}{n} = \frac{p}{q}$, *then* $a^{\frac{m}{n}} = a^{\frac{p}{q}}$.
c. *If r and s are rational numbers, then:*
   i. $a^r a^s = a^{r+s}$ *and* $\frac{a^r}{a^s} = a^{r-s}$,
   ii. $(ab)^r = a^r b^r$ *and* $(\frac{a}{b})^r = \frac{a^r}{b^r}$, *and*
   iii. $(a^r)^s = a^{rs}$.

Solution. It should be noticed first that $(a^n)^m = (a^m)^n = a^{mn}$ for all $a \in \mathbb{R}$ and all natural numbers $m, n \in \mathbb{N}$.

(a) Notice that

$$\begin{aligned}[(\sqrt[n]{a})^m]^n &= \underbrace{(\sqrt[n]{a})^m \cdot (\sqrt[n]{a})^m \cdots (\sqrt[n]{a})^m}_{n\text{-factors}} = \underbrace{\sqrt[n]{a} \cdot \sqrt[n]{a} \cdots \sqrt[n]{a}}_{mn\text{-factors}} \\ &= \underbrace{(\sqrt[n]{a})^n \cdot (\sqrt[n]{a})^n \cdots (\sqrt[n]{a})^n}_{m\text{-factors}} = \underbrace{a \cdot a \cdots a}_{m\text{-factors}} = a^m.\end{aligned}$$

Since the n^{th}-roots are unique (Theorem 3.5), we infer that $(\sqrt[n]{a}\,)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$.

(b) Assume $m, n, p, q \in \mathbb{N}$ satisfy $\frac{m}{n} = \frac{p}{q}$, or $pn = mq$. Using part (a), we see that

$$(a^{\frac{p}{q}})^n = (\sqrt[q]{a^p})^n = [(\sqrt[q]{a})^p]^n = (\sqrt[q]{a})^{pn} = (\sqrt[q]{a})^{mq} = [(\sqrt[q]{a})^q]^m = a^m,$$

and this shows that $a^{\frac{p}{q}} = a^{\frac{m}{n}}$.

(c) The formulas can be established easily if r and s are integers. Now, let r and s be rational numbers. We shall assume that r and s are also positive and leave the "negative case" for the reader. By part (b), we can also suppose that $r = \frac{m}{n}$ and $s = \frac{p}{n}$. Since $(\sqrt[n]{a} \cdot \sqrt[n]{b})^n = (\sqrt[n]{a})^n \cdot (\sqrt[n]{b})^n = ab$, we see that $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$. Now note that

i. $a^r a^s = \sqrt[n]{a^m} \cdot \sqrt[n]{a^p} = \sqrt[n]{a^m a^p} = \sqrt[n]{a^{m+p}} = a^{\frac{m+p}{n}} = a^{r+s}$.
ii. $(ab)^r = \sqrt[n]{(ab)^m} = \sqrt[n]{a^m b^m} = (\sqrt[n]{a^m})(\sqrt[n]{b^m}) = a^r b^r$.
iii. $(a^r)^s = \sqrt[n]{(\sqrt[n]{a^m})^p} = \sqrt[n]{\sqrt[n]{a^{pm}}} = \sqrt[n^2]{a^{pm}} = a^{\frac{pm}{n^2}} = a^{rs}$.

We leave the remaining cases for the reader.

4. SEQUENCES OF REAL NUMBERS

Problem 4.1. *Show that if* $|x| < 1$, *then* $\lim x^n = 0$.

Solution. Let $x_n = |x|^n$ for each n. Then $x_{n+1} = |x|x_n$ holds for each n, and the assumption $|x| < 1$ implies $0 \le x_{n+1} \le x_n$. By Theorem 4.3, $a = \lim x_n$ exists. It follows that $a = a|x|$ (or $(1 - |x|)a = 0$) must hold, and from this that $a = 0$.

A direct way of proving that $\lim x^n = 0$ goes as follows. Observe first that we can suppose $0 < x < 1$. Now, if $\epsilon > 0$ is given, then note that

$$x^n < \epsilon \iff \ln(x^n) = n \ln x < \ln \epsilon \iff n > \frac{\ln \epsilon}{\ln x}.$$

Problem 4.2. *Show that* $\lim x_n = x$ *holds if and only if every subsequence of* $\{x_n\}$ *has a subsequence that converges to* x.

Solution. If $\lim x_n = x$, then every subsequence must converge to x. So, every subsequence of a subsequence (as being itself a subsequence of $\{x_n\}$) must converge to x.

For the converse, assume that each subsequence of $\{x_n\}$ has a subsequence that converges to x. Now, suppose by way of contradiction that $\{x_n\}$ does not converge to x. Then for some $\varepsilon > 0$ we must have $|x - x_n| \ge \varepsilon$ for an infinite number of n. So, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $|x - y_n| \ge \varepsilon$ for each n. However, the latter contradicts the fact that $\{y_n\}$ has a subsequence that converges to x. Therefore, $\lim x_n = x$.

Problem 4.3. *Consider two sequences $\{k_n\}$ and $\{m_n\}$ of strictly increasing natural numbers such that for some $\ell \in \mathbb{N}$ we have*

$$\{\ell, \ell+1, \ell+2, \ldots\} \subseteq \{k_1, k_2, \ldots\} \cup \{m_1, m_2, \ldots\}.$$

Show that a sequence of real numbers $\{x_n\}$ converges in $\mathbb{R}$ if and only if both subsequences $\{x_{k_n}\}$ and $\{x_{m_n}\}$ of $\{x_n\}$ converge in $\mathbb{R}$ and they satisfy $\lim x_{k_n} = \lim x_{m_n}$ *(in which case the common limit is also the limit of the sequence).*

In particular, show that a sequence of real numbers $\{x_n\}$ converges in $\mathbb{R}$ if and only if the "even" and "odd" subsequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ both converge in $\mathbb{R}$ and they satisfy $\lim x_{2n} = \lim x_{2n-1}$.

Solution. If $x_n \to x$, then clearly $x_{k_n} \to x$ and $x_{m_n} \to x$. For the converse, assume that $x_{k_n} \to x$ and $x_{m_n} \to x$. Let $\epsilon > 0$. Choose some $n_0 \in \mathbb{N}$ such that

$$\left|x_{k_n} - x\right| < \epsilon \quad \text{and} \quad \left|x_{m_n} - x\right| < \epsilon \quad \text{for all } n \geq n_0. \tag{$\star$}$$

Put $\ell_0 = \max\{\ell, k_{n_0}, m_{n_0}\}$, and we claim that

$$\left|x_n - x\right| < \epsilon \quad \text{for all } n \geq \ell_0.$$

To see this, let $n \geq \ell_0$. Then the assumption

$$\{\ell, \ell+1, \ell+2, \ldots\} \subseteq \{k_1, k_2, \ldots\} \cup \{m_1, m_2, \ldots\}$$

guarantees the existence of some $r \in \mathbb{N}$ such that $k_r = n$ or $m_r = n$. Since $r < n_0$ implies $k_r < k_{n_0} \leq \ell_0$ and $m_r < m_{n_0} \leq \ell_0$, we see that $r \geq n_0$. Hence, either $x_n = x_{k_r}$ of $x_n = x_{m_r}$ (with $r \geq n_0$), and so from ($\star$) it follows that $|x_n - x| < \epsilon$. This shows that $x_n \to x$.

The last part should be immediate from the above conclusion.

Problem 4.4. *Find the* lim sup *and* lim inf *for the sequence* $\{(-1)^n\}$.

Solution. We have $\liminf(-1)^n = -1$ and $\limsup(-1)^n = 1$.

Problem 4.5. *Find the* lim sup *and* lim inf *of the sequence $\{x_n\}$ defined by*

$$x_1 = \tfrac{1}{3}, \ x_{2n} = \tfrac{1}{3}x_{2n-1}, \ \textit{and} \ x_{2n+1} = \tfrac{1}{3} + x_{2n} \ \textit{for} \ n = 1, 2, \ldots.$$

Solution. We claim that

$$x_{2n} = \frac{1}{3^2}\sum_{k=0}^{n-1}\frac{1}{3^k} \quad \text{and} \quad x_{2n+1} = \frac{1}{3}\sum_{k=0}^{n}\frac{1}{3^k}$$

hold for $n = 1, 2, \ldots$. The validity of the identities can be established by induction. We shall establish the validity of the second identity and leave the

verification of the first to the reader. For $n = 1$, we have

$$x_3 = x_{2\cdot 1+1} = \tfrac{1}{3} + x_2 = \tfrac{1}{3} + \tfrac{1}{3}x_1 = \tfrac{1}{3} + \tfrac{1}{9} = \tfrac{1}{3}\left(1 + \tfrac{1}{3}\right) = \tfrac{1}{3}\sum_{k=0}^{1}\tfrac{1}{3^k}.$$

Now, assume that $x_{2n+1} = \frac{1}{3}\sum_{k=0}^{n}\frac{1}{3^k}$ holds for some n. Then,

$$x_{2(n+1)+1} = \tfrac{1}{3} + x_{2(n+1)} = \tfrac{1}{3} + \tfrac{1}{3}x_{2n+1} = \tfrac{1}{3} + \tfrac{1}{9}\sum_{k=0}^{n}\tfrac{1}{3^k} = \tfrac{1}{3}\sum_{k=0}^{n+1}\tfrac{1}{3^k},$$

and the induction is complete. Consequently,

$$\lim_{n\to\infty} x_{2n} = \tfrac{1}{3^2}\sum_{k=0}^{\infty}\tfrac{1}{3^k} = \tfrac{1}{3^2}\cdot\tfrac{3}{2} = \tfrac{1}{6} \quad\text{and}\quad \lim_{n\to\infty} x_{2n+1} = \tfrac{1}{3}\sum_{k=0}^{\infty}\tfrac{1}{3^k} = \tfrac{1}{2}.$$

Now, we claim that $\frac{1}{6}$ and $\frac{1}{2}$ are the only limit points of $\{x_n\}$. To see this, let a be a real number different from $\frac{1}{6}$ and $\frac{1}{2}$. Pick some $\varepsilon > 0$ such that

$$(a-\varepsilon, a+\varepsilon)\cap\left(\tfrac{1}{6}-\varepsilon, \tfrac{1}{6}+\varepsilon\right) = \emptyset \quad\text{and}\quad (a-\varepsilon, a+\varepsilon)\cap\left(\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon\right) = \emptyset.$$

Next, note that there exists some k such that $x_{2n} \in \left(\frac{1}{6}-\varepsilon, \frac{1}{6}+\varepsilon\right)$ and $x_{2n+1} \in \left(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right)$ hold for all $n \geq k$. Therefore, $|x_n - a| \geq \varepsilon$ holds for all $n \geq k$, and this shows that a cannot be a limit point of the sequence $\{x_n\}$. Consequently,

$$\liminf x_n = \tfrac{1}{6} \quad\text{and}\quad \limsup x_n = \tfrac{1}{2}.$$

Problem 4.6. *Let $\{x_n\}$ be a bounded sequence. Show that*

$$\limsup(-x_n) = -\liminf x_n \quad \textit{and} \quad \liminf(-x_n) = -\limsup x_n.$$

Solution. We shall use the fact that $\limsup x_n$ and $\liminf x_n$ are the largest and smallest limit points of $\{x_n\}$, respectively. We shall establish the first formula.

Choose two subsequences $\{y_n\}$ and $\{z_n\}$ of $\{x_n\}$ such that $\lim y_n = \liminf x_n$ and $\lim(-z_n) = \limsup(-x_n)$. Then

$$\begin{aligned} -\liminf x_n &= \lim(-y_n) \\ &\leq \limsup(-x_n) = \lim(-z_n) = -\lim z_n \\ &\leq -\liminf x_n, \end{aligned}$$

and so $\limsup(-x_n) = -\liminf x_n$.

Problem 4.7. *If $\{x_n\}$ and $\{y_n\}$ are two bounded sequences, then show that*

a. $\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$, *and*
b. $\liminf(x_n + y_n) \ge \liminf x_n + \liminf y_n$.

Moreover, show that if one of the sequences converges, then equality holds in both (a) and (b).

Solution. (a) By passing to a subsequence, we can assume that $\lim(x_n + y_n) = \limsup(x_n + y_n)$. Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{k_n}\}$ that converges. Let $x = \lim x_{k_n}$. By the same reasoning, there exists a subsequence of $\{y_{k_n}\}$ that converges to some y. Thus, there exists a strictly increasing sequence $\{m_n\}$ of natural numbers such that $x = \lim x_{m_n}$ and $y = \lim y_{m_n}$. Hence,

$$\limsup(x_n + y_n) = x + y = \lim x_{m_n} + \lim y_{m_n} \le \limsup x_n + \limsup y_n.$$

Finally, if $x = \lim x_n$ holds, then pick a subsequence $\{y_{k_n}\}$ of $\{y_n\}$ such that $\lim y_{k_n} = \limsup y_n$, and note that

$$\begin{aligned}\limsup x_n + \limsup y_n &= x + \lim y_{k_n} \\ &= \lim(x_{k_n} + y_{k_n}) \le \limsup(x_n + y_n).\end{aligned}$$

(b) It follows from (a) by using the preceding problem.

Problem 4.8. *Prove that the* lim sup *and* lim inf *processes "preserve inequalities." That is, show that if two bounded sequences $\{x_n\}$ and $\{y_n\}$ of real numbers satisfy $x_n \le y_n$ for all $n \ge n_0$, then*

$$\liminf x_n \le \liminf y_n \quad \textit{and} \quad \limsup x_n \le \limsup y_n.$$

Solution. First, we shall show that if two sequences of real numbers $\{s_n\}$ and $\{t_n\}$ converge in $\mathbf{R}$ (say $s_n \to s$ and $t_n \to t$) and $s_n \le t_n$ for each $n \ge n_0$, then $s \le t$.

Indeed if, $s > t$ is true, then let $\epsilon = \frac{s-t}{2} > 0$ and note that for all n sufficiently large, we must have

$$s_n \in (s-\epsilon, s+\epsilon) = \left(\tfrac{s+t}{2}, \tfrac{3s-t}{2}\right) \quad \text{and} \quad t_n \in (t-\epsilon, t+\epsilon) = \left(\tfrac{3t-s}{2}, \tfrac{s+t}{2}\right).$$

That is, $t_n < \frac{s+t}{2} < s_n$ must hold for all n sufficiently large, which is impossible. Hence, $s \le t$.

Now, assume that two bounded sequences of real numbers $\{x_n\}$ and $\{y_n\}$ satisfy $x_n \le y_n$ for all $n \ge n_0$. Put

$$s_n = \inf_{k \ge n} x_k \quad \text{and} \quad t_n = \inf_{k \ge n} y_k .$$

If $n \ge n_0$, then notice that for each $r \ge n$ we have $s_n = \inf_{k \ge n_0} x_k \le x_r \le y_r$, and so $s_n \le \inf_{r \ge n} y_r = t_n$ for each $n \ge n_0$. By the discussion of the first part, we infer that

$$\liminf x_n = \lim s_n \le \lim t_n = \liminf y_n .$$

The lim sup case can be established in a similar manner, or by using the formula $\limsup x_n = -\liminf(-x_n)$.

Problem 4.9. *Show that* $\lim \sqrt[n]{n} = 1$ (*and conclude from this that* $\lim \sqrt[n]{a} = 1$ *for each* $a > 0$).

Solution. Note that $\sqrt[n]{n} = \left(\sqrt[n]{\sqrt{n}}\right)^2$. An easy inductive argument shows that $\sqrt[n]{\sqrt{n}} > 1$ holds for each n. Thus, we can write $\sqrt[n]{\sqrt{n}} = 1 + x_n$ with $x_n > 0$. Since $(1+a)^n \ge 1 + na$ holds for each n and each $a \ge 0$ (see Problem 2.13), we get

$$\sqrt{n} = \left(\sqrt[n]{\sqrt{n}}\right)^n = (1+x_n)^n \ge 1 + nx_n ,$$

and so $0 < x_n \le \frac{1}{\sqrt{n}} - \frac{1}{n}$. This implies $\lim x_n = 0$. Therefore,

$$\sqrt[n]{n} = \left(\sqrt[n]{\sqrt{n}}\right)^2 = (1+x_n)^2 \longrightarrow 1.$$

An alternate proof goes as follows: By L'Hôpital's Rule, we have $\lim_{x\to\infty} \frac{\ln x}{x} = 0$, and so $\lim_{n\to\infty} \frac{\ln n}{n} = 0$. Therefore, using that the exponential function is continuous, we infer that

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^0 = 1.$$

For the parenthetical part, assume first $a > 1$. Then it is easy to see that $1 \le \sqrt[n]{a} \le \sqrt[n]{n}$ holds true for all $n > a$. Consequently, by the "Sandwich Theorem," we see that $\lim \sqrt[n]{a} = 1$. If $0 < a < 1$, then $\frac{1}{a} > 1$, and so $\lim \sqrt[n]{\frac{1}{a}} = \lim \frac{1}{\sqrt[n]{a}} = 1$, from which it follows that $\lim \sqrt[n]{a} = 1$ holds true in this case, too.

Problem 4.10. *If $\{x_n\}$ is a sequence of strictly positive real numbers, then show that*

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \le \liminf_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}.$$

Conclude from this that if $\lim \frac{x_{n+1}}{x_n}$ *exists in* $\mathbb{R}$, *then* $\lim \sqrt[n]{x_n}$ *also exists and* $\lim \sqrt[n]{x_n} = \lim \frac{x_{n+1}}{x_n}$.

Solution. Let $\{x_n\}$ be a sequence of real numbers such that $x_n > 0$ holds for each n. We shall establish $\limsup \sqrt[n]{x_n} \le \limsup \frac{x_{n+1}}{x_n}$ and leave the similar proof of the other inequality for the reader. Put

$$x = \limsup \frac{x_{n+1}}{x_n} = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} \frac{x_{k+1}}{x_k},$$

and note that if $x = \infty$, then there is nothing to prove. So, we can assume $x < \infty$.

Let $\varepsilon > 0$ be fixed. Then there exists some k such that $\frac{x_{n+1}}{x_n} < x + \varepsilon$ holds for all $n \ge k$. Now, for $n > k$ we have

$$x_n = \tfrac{x_n}{x_{n-1}} \cdot \tfrac{x_{n-1}}{x_{n-2}} \cdots \tfrac{x_{k+1}}{x_k} \cdot x_k \le (x+\varepsilon)^{n-k} x_k = (x+\varepsilon)^n c,$$

where $c = x_k(x+\varepsilon)^{-k}$ is a constant. Therefore, $\sqrt[n]{x_n} \le (x+\varepsilon)\sqrt[n]{c}$ holds for each $n \ge k$ and so, in view of $\lim \sqrt[n]{c} = 1$ (see Problem 4.9) and Problem 4.8, we infer that

$$\limsup_{n\to\infty} \sqrt[n]{x_n} \le \limsup_{n\to\infty} (x+\varepsilon)\sqrt[n]{c} = (x+\varepsilon) \lim_{n\to\infty} \sqrt[n]{c} = x + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $\limsup \sqrt[n]{x_n} \le x = \limsup \frac{x_{n+1}}{x_n}$.

Problem 4.11. *The* **sequence of averages** *of a sequence of real numbers* $\{x_n\}$ *is the sequence* $\{a_n\}$ *defined by* $a_n = \frac{x_1+x_2+\cdots+x_n}{n}$. *If* $\{x_n\}$ *is a bounded sequence of real numbers, then show that*

$$\liminf x_n \le \liminf a_n \le \limsup a_n \le \limsup x_n.$$

In particular, if $x_n \to x$, *then show that* $a_n \to x$. *Does the convergence of* $\{a_n\}$ *imply the convergence of* $\{x_n\}$?

Solution. The solution will be based upon the following properties of lim sup and lim inf:

- *If $\{u_n\}$ is a bounded sequence of real numbers, then for each $\epsilon > 0$ the inequalities*

$$u_k \geq \limsup u_n + \epsilon \quad \textit{and} \quad u_m \leq \liminf u_n - \epsilon$$

hold for finitely many k and finitely many m.

To see this, assume by way of contradiction that $u_k \geq \limsup u_n + \epsilon$ holds true for infinitely many k. Then there exists a subsequence $\{v_n\}$ of $\{u_n\}$ satisfying $v_n \geq \limsup u_n + \epsilon$ for each n. Since $\{v_n\}$ is a bounded sequence, there exists a subsequence $\{w_n\}$ of $\{v_n\}$ (and hence of $\{u_n\}$) satisfying $w_n \to w \in \mathbb{R}$. By Problem 4.8, we know that $w \geq \limsup u_n + \epsilon$, i.e., w is a limit point of $\{u_n\}$ which is greater than the largest limit point ($\limsup u_n$) of $\{u_n\}$, a contradiction.

Now, let $\{x_n\}$ be a bounded sequence of real numbers and fix $\epsilon > 0$. Put $\ell = \limsup x_n$ and let $K = \{k \in \mathbb{N}\colon x_k \geq \ell + \epsilon\}$. By the above discussion, K is a finite set. Put

$$S_n = \{i \in \mathbb{N}\colon i \in K \text{ and } i \leq n\} \quad \text{and} \quad T_n = \{i \in \mathbb{N}\colon i \notin K \text{ and } i \leq n\},$$

and define the sequences $\{s_n\}$ and $\{t_n\}$ by

$$s_n = \sum_{i \in S_n} x_i \quad \text{and} \quad t_n = \sum_{i \in T_n} x_i.$$

Clearly, $\{s_n\}$ is an eventually constant sequence, $t_n \leq n(\ell + \epsilon)$ holds for each n and $a_n = \frac{s_n}{n} + \frac{t_n}{n}$. Since $s_n/n \to 0$ and $t_n/n \leq \ell + \epsilon$ for each n, it follows from Problems 4.7 and 4.8 that

$$\limsup a_n = \limsup\left(\tfrac{s_n}{n} + \tfrac{t_n}{n}\right) = \lim \tfrac{s_n}{n} + \limsup \tfrac{t_n}{n} = \limsup \tfrac{t_n}{n} \leq \ell + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $\limsup a_n \leq \ell = \limsup x_n$. Similarly, $\liminf x_n \leq \liminf a_n$. If $x_n \to x$, then $x = \liminf x_n = \limsup x_n$, and so $x = \liminf a_n = \limsup a_n$. This implies $a_n \to x$.

The convergence of the sequence $\{a_n\}$ of averages does not imply the convergence of $\{x_n\}$. For instance, if $x_n = (-1)^n$, then $a_n \to 0$ while $\{x_n\}$ fails to converge.

Problem 4.12. *For a sequence of real numbers $\{x_n\}$ establish the following:*

a. *If $x_{n+1} - x_n \to x$ in $\mathbb{R}$, then $x_n/n \to x$.*

b. *If $\{x_n\}$ is bounded and $2x_n \le x_{n+1} + x_{n-1}$ holds for all $n = 2, 3, \ldots$, then $x_{n+1} - x_n \uparrow 0$.*

Solution. (a) Assume that $x_{n+1} - x_n \to x$ in $\mathbb{R}$. Notice that $\sum_{i=1}^{n}(x_{i+1} - x_i) = x_{n+1} - x_1$ for each n. By Problem 4.11, we have

$$\frac{1}{n}\sum_{i=1}^{n}(x_{i+1} - x_i) = \frac{1}{n}\big(x_{n+1} - x_1\big) \to x.$$

Since $x_1/n \to 0$, it follows that $x_{n+1}/n \to x$. Now note that

$$\frac{x_n}{n} = \frac{x_n}{n-1} \cdot \frac{n-1}{n} \longrightarrow x \cdot 1 = x.$$

(b) The condition $2x_n \le x_{n+1} + x_{n-1}$ can be rewritten as $x_n - x_{n-1} \le x_{n+1} - x_n$ for each $n = 2, 3, \ldots$, which implies that the bounded sequence $\{x_{n+1} - x_n\}$ is an increasing sequence, and hence convergent. Let $x_{n+1} - x_n \uparrow x$ in $\mathbb{R}$. By part (a), we have $x_n/n \to x$. But, since $\{x_n\}$ is a bounded sequence, $x_n/n \to 0$. Therefore, $x = 0$, and so $x_{n+1} - x_n \uparrow 0$.

Problem 4.13. *Consider the sequence $\{x_n\}$ defined by $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n = 1, 2, \ldots$. Show that $x_n \downarrow 0$. Also, show that $\frac{x_{n+1}}{x_n} \to \frac{1}{2}$.*

Solution. We claim that

$$0 < x_{n+1} < x_n < 1 \qquad (\star)$$

holds for each $n = 1, 2, \ldots$. To verify this claim, we use induction. Since $0 < x_1 < 1$, we have $0 < 1 - x_1 < 1$, and so $0 < 1 - x_1 < \sqrt{1 - x_1} < 1$. Hence, $0 < 1 - \sqrt{1 - x_1} = x_2 < x_1 < 1$. That is, $(\star)$ is true for $n = 1$.

For the inductive argument, assume that $(\star)$ is true for some n. This implies $0 < 1 - x_n < 1 - x_{n+1} < 1$, and so $0 < \sqrt{1 - x_n} < \sqrt{1 - x_{n+1}} < 1$, from which it follows that

$$0 < x_{n+2} = 1 - \sqrt{1 - x_{n+1}} < 1 - \sqrt{1 - x_n} = x_{n+1} < 1,$$

which shows that $(\star)$ is true for $n + 1$. This completes the induction and guarantees that $(\star)$ is true for each n.

Now, since $\{x_n\}$ is decreasing and bounded from below, it converges, say to $x \in \mathbb{R}$. Clearly, $0 \le x < 1$. Moreover, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\big(1 - \sqrt{1 - x_n}\,\big) = 1 - \sqrt{1 - x}.$$

In other words, x is the non-negative solution of the equation $x = 1 - \sqrt{1 - x}$. Solving the equation yields $x = 0$ or $x = 1$. Hence, $x = 0$, and so $x_n \downarrow 0$.

For the last part, notice that

$$\frac{x_{n+1}}{x_n} = \frac{1-\sqrt{1-x_n}}{x_n} = \frac{1}{1+\sqrt{1-x_n}} \longrightarrow \frac{1}{2},$$

and the solution is complete.

Problem 4.14. *Show that the sequence $\{x_n\}$ defined by*

$$x_n = \left(1+\frac{1}{n}\right)^n$$

is a convergent sequence.

Solution. From the binomial expansion:

$$\begin{aligned} x_n &= (1+\tfrac{1}{n})^n = \sum_{i=0}^{n} \tbinom{n}{i}\tfrac{1}{n^i} = 1+\sum_{i=1}^{n} \tbinom{n}{i}\tfrac{1}{n^i} \\ &= 1+\sum_{i=1}^{n} \tfrac{n(n-1)\cdots(n-i+1)}{i!}\cdot\tfrac{1}{n^i} \\ &= 1+\sum_{i=1}^{n} \tfrac{1}{i!}(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots(1-\tfrac{i-1}{n}) \\ &\le 1+\sum_{i=1}^{n+1} \tfrac{1}{i!}(1-\tfrac{1}{n+1})(1-\tfrac{2}{n+1})\cdots(1-\tfrac{i-1}{n+1}) \\ &= (1+\tfrac{1}{n+1})^{n+1} = x_{n+1}. \end{aligned}$$

Thus, $x_n \uparrow$ holds. Also, note that for $n \ge 2$ we have

$$x_n = 1+\sum_{i=1}^{n} \tfrac{1}{i!}(1-\tfrac{1}{n})\cdots(1-\tfrac{i-1}{n}) \le 2+\sum_{i=2}^{n}\tfrac{1}{i!} \le 2+\sum_{i=2}^{n}\tfrac{1}{2^{i-1}} \le 3.$$

By Theorem 4.3, $\{x_n\}$ converges. (Of course, $\lim x_n = e = 2.718\cdots$.)

Problem 4.15. *Assume that a sequence $\{x_n\}$ satisfies*

$$|x_{n+1}-x_n| \le \alpha|x_n - x_{n-1}|$$

for $n = 2, 3, \ldots$ and some fixed $0 < \alpha < 1$. Show that $\{x_n\}$ is a convergent sequence.

Solution. Let $c = |x_2 - x_1|$. An easy inductive argument shows that for each n we have $|x_{n+1} - x_n| \le c\alpha^{n-1}$. Thus,

$$|x_{n+p} - x_n| \le \sum_{i=1}^{p} |x_{n+i} - x_{n+i-1}| \le c \sum_{i=1}^{p} \alpha^{n+i-2} \le \tfrac{c}{1-\alpha}\alpha^{n-1}$$

holds for all n and all p. Since $\lim \alpha^n = 0$, it follows that $\{x_n\}$ is a Cauchy sequence, and hence, a convergent sequence.

Problem 4.16. *Show that the sequence $\{x_n\}$, defined by*

$$x_1 = 1 \text{ and } x_{n+1} = \frac{1}{3 + x_n} \text{ for } n = 1, 2, \dots,$$

converges and determine its limit.

Solution. Clearly, $x_n > 0$ holds for each n. Now, note that

$$|x_{n+1} - x_n| = \left| \tfrac{1}{3+x_n} - \tfrac{1}{3+x_{n-1}} \right| = \tfrac{|x_n - x_{n-1}|}{(3+x_n)(3+x_{n-1})} \le \tfrac{1}{9}|x_n - x_{n-1}|$$

holds for $n = 2, 3, \dots$. By Problem 4.15, the sequence $\{x_n\}$ converges. If $\lim x_n = x$, then $x \ge 0$ and

$$x = \lim x_{n+1} = \frac{1}{3 + \lim x_n} = \frac{1}{3 + x}.$$

Solving the equation, we get $x = \frac{-3+\sqrt{13}}{2}$.

Problem 4.17. *Consider the sequence $\{x_n\}$ of real numbers defined by $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{1+x_n}$ for $n = 1, 2, \dots$. Show that $\{x_n\}$ is a convergent sequence and that* $\lim x_n = \sqrt{2}$.

Solution. An easy inductive argument shows that $x_n > 0$ for each n. This implies that, in fact, we have $1 \le x_n \le 2$ for each n. Now, note that

$$\begin{aligned}
|x_{n+1} - x_n| &= \left| \frac{1}{1 + x_n} - \frac{1}{1 + x_{n-1}} \right| \\
&= \frac{|x_n - x_{n-1}|}{(1 + x_n)(1 + x_{n-1})} \\
&\le \frac{|x_n - x_{n-1}|}{(1+1)(1+1)} = \tfrac{1}{4}|x_n - x_{n-1}|
\end{aligned}$$

for each $n = 2, 3, \dots$. By Problem 4.15, the sequence $\{x_n\}$ converges. Let

$x_n \to x$. Since $x_n \geq 1$ for each n, we see that $x \geq 1$. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(1 + \frac{1}{1+x_n}\right) = 1 + \frac{1}{1+x}.$$

That is, x is the positive solution of the equation $x = 1+\frac{1}{1+x}$, or $x^2+x = 1+x+1$. This implies $x^2 = 2$, and so $x = \sqrt{2}$.

Problem 4.18. *Define the sequence* $\{x_n\}$ *by* $x_1 = 1$ *and*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right), \quad n = 1, 2, \ldots.$$

Show that $\{x_n\}$ *converges and that* $\lim x_n = \sqrt{2}$.

Solution. Clearly, $x_n > 0$ holds for each n. (Use induction to prove this!) Also,

$$x_{n+1}^2 - 2 = \tfrac{1}{4}\left(x_n - \tfrac{2}{x_n}\right)^2 \geq 0$$

holds for each n. Thus, if $n \geq 2$, then

$$x_{n+1} - x_n = \tfrac{1}{2}\left(x_n + \tfrac{2}{x_n}\right) - x_n = \tfrac{2-x_n^2}{2x_n} \leq 0,$$

and so $0 < x_{n+1} < x_n$ holds for each $n \geq 2$. By Theorem 4.3, $x = \lim x_n$ exists. Since $x_n^2 \geq 2$ holds for each $n \geq 2$, we see that $x > 0$. From the recursive formula, it follows that $2x = x + \frac{2}{x}$, or $x^2 = 2$. (Note also that the limit is independent of the initial choice $x_1 > 0$.)

Problem 4.19. *Define the sequence* $x_n = \sum_{k=1}^{n} \frac{1}{k}$ *for* $n = 1, 2, \ldots$. *Show that* $\{x_n\}$ *does not converge in* $\mathbb{R}$. *(See also Problem 5.10.)*

Solution. The inequality

$$\begin{aligned} x_{2n} - x_n &= \tfrac{1}{n+1} + \tfrac{1}{n+2} + \cdots + \tfrac{1}{n+n} \\ &\geq \tfrac{1}{2n} + \tfrac{1}{2n} + \cdots + \tfrac{1}{2n} = n \cdot \tfrac{1}{2n} = \tfrac{1}{2} \end{aligned}$$

shows that $\{x_n\}$ is not a Cauchy sequence, and hence, is not convergent in $\mathbb{R}$.

Problem 4.20. *Let* $-\infty < a < b < \infty$ *and* $0 < \lambda < 1$. *Define the sequence*

$\{x_n\}$ by $x_1 = a$, $x_2 = b$ and

$$x_{n+2} = \lambda x_n + (1-\lambda)x_{n+1} \text{ for } n = 1, 2, \ldots.$$

Show that $\{x_n\}$ converges in $\mathbb{R}$ and find its limit.

Solution. Rewriting $x_{n+2} = \lambda x_n + (1-\lambda)x_{n+1} = \lambda x_n + x_{n+1} - \lambda x_{n+1}$ in the form $x_{n+2} - x_{n+1} = \lambda(x_n - x_{n+1})$, we see that

$$|x_{n+2} - x_{n+1}| = \lambda|x_{n+1} - x_n|$$

holds for each n. Now, a glance at Problem 4.15 guarantees that $\{x_n\}$ is a convergent sequence. However, we cannot get the limit of the sequence $\{x_n\}$ by taking limits in both sides of the recursive formula $x_{n+2} = \lambda x_n + (1-\lambda)x_{n+1}$. We shall compute the limit of the sequence $\{x_n\}$ using a different method.

For simplicity put $\mu = 1 - \lambda$. First, we shall verify that

$$x_1 < x_3 < \cdots < x_{2n+1} < x_{2n} < x_{2n-2} < \cdots < x_2$$

holds for each n.

The proof is by induction. For $n = 1$, the inequalities reduce to $x_1 < x_2$ which is obviously true. So, for the inductive step, assume $x_{2n-1} < x_{2n}$ for some n. Then

$$x_{2n+1} = \lambda x_{2n-1} + \mu x_{2n} = x_{2n-1} + \mu(x_{2n} - x_{2n-1}) > x_{2n-1}$$

and

$$x_{2n+1} = \lambda x_{2n-1} + \mu x_{2n} = x_{2n} - \lambda(x_{2n} - x_{2n-1}) < x_{2n}.$$

Now, note that

$$x_{2n+1} < \lambda x_{2n-1} + (1-\lambda)x_{2n} = x_{2n+2} < x_{2n}.$$

Next, if we let $d_n = x_{2n} - x_{2n-1}$, then it is easily to verify (see Figure 1.1) that

$$\begin{aligned} d_{n+1} &= \lambda\mu d_n, && (1)\\ x_{2n+1} &= x_{2n-1} + \mu d_n, \quad \text{and} && (2)\\ x_{2n+2} &= x_{2n} - \lambda^2 d_n. && (3) \end{aligned}$$

From (1) it follows that

$$d_n = (\lambda\mu)^{n-1}d_1 = (\lambda\mu)^{n-1}(b-a),$$

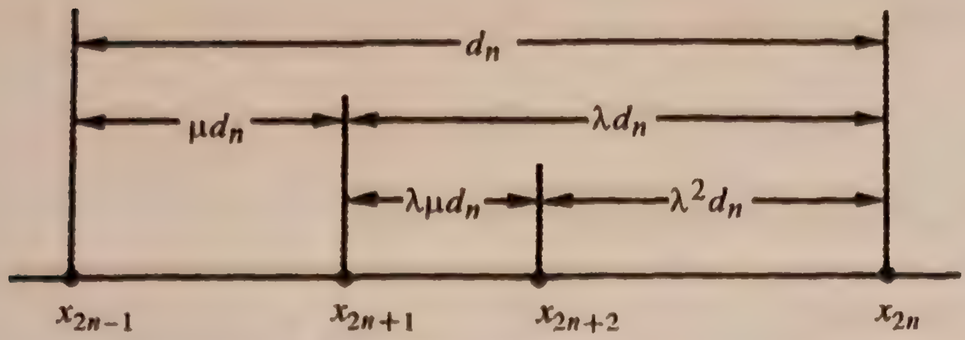

FIGURE 1.1.

and so from (2) and (3), we obtain

$$\begin{aligned}
x_{2n+1} &= x_1 + \sum_{i=1}^{n}(x_{2i+1} - x_{2i-1}) = x_1 + \sum_{i=1}^{n} \mu d_i \\
&= x_1 + \mu \sum_{i=1}^{n}(\lambda\mu)^{i-1} d_1 = x_1 + \frac{\mu d_1[1-(\lambda\mu)^n]}{1-\lambda\mu} \\
&= a + \frac{\mu(b-a)[1-(\lambda\mu)^n]}{1-\lambda\mu},
\end{aligned}$$

and

$$\begin{aligned}
x_{2n+2} &= x_2 - \sum_{i=1}^{n}(x_{2i} - x_{2i+2}) = x_2 - \lambda^2 \sum_{i=1}^{n}(\lambda\mu)^{i-1} d_i \\
&= x_2 - \frac{\lambda^2[1-(\lambda\mu)^n]d_1}{1-\lambda\mu} = b - \frac{\lambda^2(b-a)[1-(\lambda\mu)^n]}{1-\lambda\mu}.
\end{aligned}$$

Therefore,

$$x_{2n+1} \uparrow a + \frac{\mu(b-a)}{1-\lambda\mu} = \frac{\lambda^2 a + \mu b}{1-\lambda\mu} \quad \text{and} \quad x_{2n} \downarrow b - \frac{\lambda^2(b-a)}{1-\lambda\mu} = \frac{\lambda^2 a + \mu b}{1-\lambda\mu},$$

and consequently, $\lim x_n = \frac{\lambda^2 a + \mu b}{1-\lambda\mu}$.

Problem 4.21. *Let G be a nonempty subset of $\mathbf{R}$, which is a group under addition (i.e., if $x, y \in G$, then $x + y \in G$ and $-x \in G$). Show that between any two distinct real numbers there exists an element of G or else there exists $a \in \mathbf{R}$ such that $G = \{na\colon n = 0, \pm1, \pm2, \ldots\}$.*

Solution. Assume $G \neq \{0\}$. Let $a = \inf G \cap (0, \infty)$. We distinguish two cases. (1) $a > 0$. *In this case, we shall show that $G = \{na\colon n = 0, \pm1, \ldots\}$.*

To see this, note first that $a \in G$. Indeed, if $a \notin G$, then there exist $x, y \in G$ with $a < x < y < \frac{3a}{2}$. Then, the element $z = y - x \in G$ satisfies $0 < z < \frac{a}{2}$, contradicting the definition of a. Now, if $x \in G$, then $na \leq x < (n+1)a$ must hold for some integer n. However, $x = na$ must also hold, since otherwise the element $x - na \in G$ satisfies $0 < x - na < a$, which is again a contradiction.
(2) $a = 0$. *In this case, we claim that between any two distinct real numbers there is an element of* G.

To see this, we only need to consider $0 < x < y$. Let $\delta = \min\{x, y - x\} > 0$. Choose some element $z \in G$ with $0 < z < \delta$. By the Archimedean property, the set $A = \{n \in \mathbf{N}\colon nz \geq y\}$ is nonempty, and by the Well Ordering Principle the element $k = \min A$ exists. Now, note that the element $b = (k-1)z \in G$ satisfies $x < b < y$.

Problem 4.22. *Determine the limit points of the sequence* $\{\cos n\}$.

Solution. We claim that the set of limit points of the sequence $\{\cos n\}$ is $[-1, 1]$. To prove this, we shall need two facts from elementary calculus.
a) The Intermediate Value Theorem; and
b) The inequality $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbf{R}$.

Let $G = \{n + 2m\pi\colon\ n, m \text{ integers}\}$. Clearly, G is a group under addition, and since π is an irrational number, it is easy to see that the group G is not of the form $\{na\colon\ n = 0, \pm 1, \pm 2, \ldots\}$. Now, let $x \in [-1, 1]$ and let $\varepsilon > 0$. By the Intermediate Value Theorem, there exists some $y \in \mathbf{R}$ satisfying $\cos y = x$. The preceding Problem 4.21 shows that there exist two integers n and m satisfying $y < n + 2m\pi < y + \varepsilon$. Thus,

$$\left|x - \cos n\right| = \left|\cos y - \cos(n + 2m\pi)\right| \leq n + 2m\pi - y < \varepsilon.$$

The above arguments show that given $x \in [-1, 1]$ and $\varepsilon > 0$, there exists some non-negative integer n with $|x - \cos n| < \varepsilon$. From this, it easily follows (how?) that every point of $[-1, 1]$ must be a limit point of $\{\cos n\}$.

Problem 4.23. *For each n define* $f_n\colon [-1, 1] \to \mathbf{R}$ *by* $f_n(x) = x^n$. *Determine* $\limsup f_n$ *and* $\liminf f_n$.

Solution. We have

$$\limsup f_n(x) = \begin{cases} 1, & \text{if } x = -1 \\ 0, & \text{if } |x| < 1 \\ 1, & \text{if } x = 1 \end{cases}$$

and

$$\liminf f_n(x) = \begin{cases} -1, & \text{if } x = -1 \\ 0, & \text{if } |x| < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

Problem 4.24. *Show that every sequence of real numbers has a monotone subsequence. Use this conclusion to provide an alternate proof of the Bolzano–Weierstrass property of the real numbers:* Every bounded sequence has a convergent subsequence. (*See Corollary 4.7.*)

Solution. Let $\{x_n\}$ be a sequence of real numbers. We consider the set of natural numbers

$$S = \{k \in \mathbf{N}\colon\ x_k \le x_m \ \text{ for all } m \ge k\},$$

and distinguish two cases.

1. *S is infinite.*

In this case, we can write $S = \{k_1, k_2, \ldots\}$ with $k_1 < k_2 < \cdots$. Now, it should be clear that the subsequence $\{x_{k_n}\}$ of $\{x_n\}$ is increasing.

2. *S is finite (and possibly empty).*

In this case, if we put $k_1 = 1 + \max S$ (let $\max S = 0$ if $S = \emptyset$), then for each $k \ge k_1$ there exists some $m > k$ such that $x_m < x_k$. So, by induction, if k_n has been chosen, then we can select some natural number k_{n+1} with $k_{n+1} > k_n$ and $x_{k_{n+1}} < x_{k_n}$. This implies that $\{x_{k_n}\}$ is a strictly decreasing subsequence of $\{x_n\}$, and the claim is established.

For the Bolzano–Weierstrass property, notice that if $\{x_n\}$ is a bounded sequence, then, by the above, $\{x_n\}$ has a monotone subsequence which (by Theorem 4.3) must be convergent in $\mathbf{R}$. (We remark that this result shows that not only a bounded sequence has a convergent subsequence but it also has a monotone convergent subsequence.)

5. THE EXTENDED REAL NUMBERS

Problem 5.1. *Let $\{x_n\}$ be a sequence of $\mathbf{R}^*$. Define a limit point of $\{x_n\}$ in $\mathbf{R}^*$ to be any element x of $\mathbf{R}^*$ for which there exists a subsequence of $\{x_n\}$ that converges to x.*

Show that

$$\limsup x_n = \inf_n \Big[\sup_{k\ge n} x_k\Big] \quad \textit{and} \quad \liminf x_n = \sup_n \Big[\inf_{k\ge n} x_k\Big]$$

are the largest and smallest limit points of $\{x_n\}$ in $\mathbf{R}^*$.

Solution. The lim sup case is established. Let $x = \limsup x_n \in \mathbf{R}^*$. Then three cases arise:

a) $x \in \mathbf{R}$.
In this case, repeat the proof of Theorem 4.6.

b) $x = \infty$.
In this case, we have only to show that x is a limit point of $\{x_n\}$. Note that $\bigvee_{i=n}^{\infty} x_i = \infty$ for each n. Choose some $k_1 \geq 1$ such that $x_{k_1} > 1$. Now, by induction: If k_n has been selected so that $x_{k_n} > n$, then use $\bigvee_{i>k_n} x_i = \infty$ to choose some $k_{n+1} \geq k_n + 1 > k_n$ so that $x_{k_{n+1}} > n+1$. Clearly, $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$ satisfying $\lim x_{k_n} = \infty$.

c) $x = -\infty$.
In this case, we shall show that $\lim x_n = -\infty$. Let $0 < M < \infty$. From $\bigvee_{i=n}^{\infty} x_i \downarrow -\infty$, it follows that $\bigvee_{i=n}^{\infty} x_i < -M$ for some n, and so $x_i < -M$ for all $i \geq n$. That is, $\lim x_n = -\infty$.

Problem 5.2. *Let $\{x_n\}$ be a sequence of positive real numbers such that $\ell = \lim \frac{x_{n+1}}{x_n}$ exists in $\mathbf{R}$. Show that:*

a. *if $\ell < 1$, then $\lim x_n = 0$, and*
b. *if $\ell > 1$, then $\lim x_n = \infty$.*

Solution. (a) Assume $\ell < 1$ and fix some δ such that $\ell < \delta < 1$; for instance, let $\delta = \frac{1+\ell}{2}$. Since $\lim \frac{x_{n+1}}{x_n} = \ell$, there exists some $k > 1$ such that $\frac{x_{n+1}}{x_n} < \delta$ holds for all $n \geq k$. Now, if $n > k$, then note that

$$\begin{aligned} x_n &= \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_{k+1}}{x_k} \cdot x_k \\ &< \underbrace{\delta \cdot \delta \cdots \delta}_{(n-k)\text{-terms}} \cdot x_k = \delta^{n-k} x_k = \Big(\frac{x_k}{\delta^k}\Big)\delta^n, \end{aligned}$$

and so if $c = \frac{x_k}{\delta^k}$, then

$$0 < x_n < c\delta^n$$

holds for all $n > k$. Since (in view of $0 < \delta < 1$) $\delta^n \to 0$, we easily infer that $x_n \to 0$.

(b) Assume now $\ell > 1$ and choose some δ such that $1 < \delta < \ell$. Since $\lim \frac{x_{n+1}}{x_n} = \ell$, there exists some $k > 1$ such that $\frac{x_{n+1}}{x_n} > \delta$ holds for all $n \geq k$. Then, as in the preceding case, there exists some constant $C > 0$ satisfying $x_n > C\delta^n$ for all $n > k$. From $\delta^n \to \infty$, it easily follows that $x_n \to \infty$.

Problem 5.3. *Let $0 \leq a_{n,m} \leq \infty$ for all m, n, and let $\sigma\colon \mathbf{N} \times \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ be*

one-to-one and onto. Show that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{n,m} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{\sigma(n,m)}.$$

Solution. It follows immediately from Theorem 5.4.

Problem 5.4. *Show that*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^2+m^2} = \infty.$$

Solution. The convergence or divergence of the series is according to the convergence or divergence of the double integral $\int_1^{\infty}\int_1^{\infty} \frac{dxdy}{x^2+y^2}$. Now, note that

$$\int_1^{\infty}\left[\int_1^{\infty} \frac{dx}{x^2+y^2}\right]dy \geq \tfrac{\pi}{2}\int_1^{\infty}\frac{dy}{y} = \infty.$$

An alternate solution goes as follows. Note first that the inequality

$$\frac{1}{n^2+m^2} > \frac{1}{(n+m)^2} > \frac{1}{(n+m)(n+m+1)} = \frac{1}{n+m} - \frac{1}{n+m+1}$$

implies $\sum_{m=1}^{\infty} \frac{1}{n^2+m^2} \geq \sum_{m=1}^{\infty}\left(\frac{1}{n+m} - \frac{1}{n+m+1}\right) = \frac{1}{n+1}$. Therefore,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^2+m^2} \geq \sum_{n=1}^{\infty}\frac{1}{n+1} = \infty.$$

Problem 5.5. *This problem describes the p-adic* **representation** *of a real number in $(0,1)$. We assume that p is a natural number such that $p \geq 2$ and $x \in (0,1)$.*

a. *Divide the interval $[0,1)$ into the p closed-open intervals*

$$[0, \tfrac{1}{p}), [\tfrac{1}{p}, \tfrac{2}{p}), \ldots, [\tfrac{p-1}{p}, 1),$$

and number them consecutively from 0 to $p-1$. Then x belongs precisely to one of these intervals, say k_1 $(0 \leq k_1 < p)$. Next, divide the interval $[\frac{k_1}{p}, \frac{k_1+1}{p})$ into p closed-open intervals (of the same length), number them consecutively from 0 to $p-1$, and let k_2 be the subinterval to which x belongs. Proceeding this way, we construct a sequence $\{k_n\}$ of non-negative

integers such that $0 \le k_n < p$ *for each* n. *Show that*

$$x = \sum_{n=1}^{\infty} \frac{k_n}{p^n}.$$

b. *Apply the same process as in* (a) *by subdividing each interval now into* p *open-closed intervals. For example, start with* $(0, 1]$ *and subdivide it into the open-closed intervals* $(0, \frac{1}{p}], (\frac{1}{p}, \frac{2}{p}], \ldots, (\frac{p-1}{p}, 1]$.

As in (*a*), *construct a sequence* $\{m_n\}$ *of non-negative integers such that* $0 \le m_n < p$ *for each* n. *Show that*

$$x = \sum_{n=1}^{\infty} \frac{m_n}{p^n}.$$

c. *Show by an example that the two sequences constructed in* (a) *and* (b) *may be different.*

In order to make the p*-adic representation of a number unique, we shall agree to take the one determined by* (a) *above. As usual, it will be written as* $x = 0.k_1k_2\cdots$.

Solution. For (a) and (b) note that $\left|x - \sum_{i=1}^{n} \frac{k_i}{p^i}\right| \le \frac{1}{p^n}$ holds for all n.

For part (c) take, for instance, $p = 2$ and note that for $x = \frac{1}{2}$ we have $k_1 = 1$ and $k_n = 0$ if $n > 1$, while $m_1 = 0$ and $m_n = 1$ for $n > 1$.

Problem 5.6. *Show that* $\mathcal{P}(\mathbb{N}) \approx \mathbb{R}$ *by establishing the following:*

i. *If* A *is an infinite set, and* $f\colon A \to B$ *is one-to-one such that* $B \setminus f(A)$ *is at-most countable, then show that* $A \approx B$.

ii. *Show that the set of numbers of* $(0, 1)$ *for which the dyadic* (*i.e.*, $p = 2$) *representation determined by* (*a*) *and* (*b*) *of the preceding exercise are different is a countable set.*

iii. *For each* $x \in (0, 1)$, *let* $x = 0.k_1k_2\cdots$ *be the dyadic representation determined by part* (*a*) *of the preceding exercise; clearly, each* k_i *is either* 0 *or* 1. *Let* $f(x) = \{n \in \mathbb{N}\colon k_n = 1\}$. *Show that* $f\colon (0, 1) \to \mathcal{P}(\mathbb{N})$ *is one-to-one such that* $\mathcal{P}(\mathbb{N}) \setminus f((0, 1))$ *is countable, and conclude from part* (*i*) *that* $(0, 1) \approx \mathcal{P}(\mathbb{N})$.

Solution. (i) Let $S = \{a_1, a_2, \ldots\}$ be a countable subset of A.

(a) Assume $B \setminus f(A) = \{b_1, \ldots, b_n\}$ is a finite set. Then $g\colon A \to B$ defined by $g(x) = f(x)$ if $x \notin S$, $g(a_i) = b_i$ for $1 \le i \le n$, and $g(a_{i+n}) = f(a_i)$ for $i = 1, 2, \ldots$ is one-to-one and onto.

(b) Assume $B \setminus f(A) = \{b_1, b_2, \ldots\}$ is countable. Then $g\colon A \to B$ defined by $g(x) = f(x)$ if $x \notin S$, $g(a_{2n+1}) = f(a_n)$ and $g(a_{2n}) = b_n$ for each n is one-to-one and onto.

(ii) Let D be the set of all numbers of $(0,1)$ for which the two sequences $\{k_n\}$ and $\{m_n\}$ determined by the preceding problem are different. Assume $x = 0.k_1k_2\cdots = 0.m_1m_2\cdots \in D$ and define the natural number $r = \min\{n\colon k_n \neq m_n\}$. We can assume $k_r = 1$ and $m_r = 0$. Then the inequalities

$$\begin{aligned} x - \sum_{n=1}^{r-1} \tfrac{m_n}{2^n} &= \sum_{n=r}^{\infty} \tfrac{m_n}{2^n} \leq \sum_{n=r+1}^{\infty} \tfrac{1}{2^n} \\ &= \tfrac{1}{2^r} \\ &\leq \tfrac{1}{2^r} + \sum_{n=r+1}^{\infty} \tfrac{k_n}{2^n} = x - \sum_{n=1}^{r-1} \tfrac{k_n}{2^n} \\ &= x - \sum_{n=1}^{r-1} \tfrac{m_n}{2^n}, \end{aligned}$$

guarantee that $x = \frac{k_1}{2} + \frac{k_2}{2^2} + \cdots + \frac{k_{r-1}}{2^{r-1}} + \frac{1}{2^r}$. In particular, note that $m_n = 1$ and $k_n = 0$ hold for each $n > r$.

On the other hand, it is not difficult to see that every x of the above type belongs to D. It is now a routine matter to verify that D is a countable set. (It is also interesting to observe that D consists precisely of the endpoints of the subintervals appearing during the construction of the expansions.)

(iii) Let $A \in \mathcal{P}(\mathbb{N})$. Define the sequence $\{m_n\}$ of $\{0,1\}$ by $m_n = 1$ if $n \in A$ and $m_n = 0$ if $n \notin A$, and then set

$$x = \sum_{n=1}^{\infty} \tfrac{m_n}{2^n}.$$

Note that $A \notin f\big((0,1)\big)$ if and only if $x \in D$. Thus, $\mathcal{P}(\mathbb{N}) \setminus f\big((0,1)\big)$ is countable, and so by part (i) and the fact that f is one-to-one, $(0,1) \approx \mathcal{P}(\mathbb{N})$ holds.

Problem 5.7. *For a sequence $\{x_n\}$ of real numbers show that the following conditions are equivalent:*

a. *The series $\sum_{n=1}^{\infty} x_n$ is rearrangement invariant in $\mathbb{R}$.*
b. *For every permutation σ of $\mathbb{N}$ the series $\sum_{n=1}^{\infty} x_{\sigma_n}$ converges in $\mathbb{R}$.*
c. *The series $\sum_{n=1}^{\infty} |x_n|$ converges in $\mathbb{R}$.*
d. *For every sequence $\{s_n\}$ of $\{-1,1\}$, the series $\sum_{n=1}^{\infty} s_n x_n$ converges in $\mathbb{R}$.*
e. *For every subsequence $\{x_{k_n}\}$ of $\{x_n\}$, the series $\sum_{n=1}^{\infty} x_{k_n}$ converges in $\mathbb{R}$.*
f. *For every $\epsilon > 0$, there exists an integer k (depending on ϵ) such that for every finite subset S of $\mathbb{N}$ with $\min S \geq k$, we have $|\sum_{n\in S} x_n| < \epsilon$.*

(Any series $\sum_{n=1}^{\infty} x_n$ satisfying any one of the above conditions is also referred to as an **unconditionally convergent series.***)*

Solution. (a)$\Longrightarrow$(b) Obvious.
(b)$\Longrightarrow$(c) Assume $\sum_{n=1}^{\infty}|x_n| = \infty$. From our hypothesis it follows that $x_n > 0$ and $x_n < 0$ both hold for infinitely many n. Split $\{x_n\}$ into two subsequences $\{y_n\}$ and $\{z_n\}$ such that $y_n \geq 0$ and $z_n < 0$ hold for all n. We can assume that $\sum_{n=1}^{\infty} y_n = \infty$.

Now, use induction to construct a strictly increasing sequence of natural numbers $\{k_n\}$ such that

1. $k_1 = 1$ and $z_1 + \sum_{i=1}^{k_1} y_i > 1$; and
2. $z_n + \sum_{i=k_n+1}^{k_{n+1}} y_i > 1$ for $n = 1, 2, \ldots$.

Then note that

$$y_1, \ldots, y_{k_1}, z_1, y_{k_1+1}, \ldots, y_{k_2}, z_2, y_{k_2+1}, \ldots$$

is a permutation of $\{x_n\}$ whose series is not convergent, contrary to our hypothesis.
(c)$\Longrightarrow$(d) Obvious.
(d)$\Longrightarrow$(e) Let $\{x_{k_n}\}$ be a subsequence of $\{x_n\}$. Put $s_i = -1$ if $i \neq k_n$ for each n, and $s_{k_n} = 1$. Then

$$\sum_{n=1}^{\infty} x_{k_n} = \tfrac{1}{2}\Big[\sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} s_n x_n\Big]$$

is a convergent series.
(e)$\Longrightarrow$(f) If (f) is false, then there exists some $\varepsilon > 0$ and a sequence $\{S_n\}$ of finite subsets of natural numbers such that $\max S_n < \min S_{n+1}$ and $\left|\sum_{i\in S_n} x_i\right| \geq \varepsilon$ hold for all n. Let

$$\bigcup_{n=1}^{\infty} S_n = \{k_1, k_2, \ldots\},$$

where $k_n \uparrow$. Then, it is easy to see that the series $\sum_{n=1}^{\infty} x_{k_n}$ does not converge in $\mathbb{R}$, contradicting (e).
(f)$\Longrightarrow$(a) Let $\sigma\colon \mathbb{N} \longrightarrow \mathbb{N}$ be a permutation. By our hypothesis, it is readily seen that the partial sums of both series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} x_{\sigma_n}$ form Cauchy sequences, and hence, both series converge in $\mathbb{R}$. Let $x = \sum_{n=1}^{\infty} x_n$ and $y = \sum_{n=1}^{\infty} x_{\sigma_n}$.

Now, if $\varepsilon > 0$ is given, then choose k so large such that

$$\Big|x - \sum_{n=1}^{r} x_n\Big| < \varepsilon, \quad \Big|y - \sum_{n=1}^{r} x_{\sigma_n}\Big| < \varepsilon, \quad \text{and} \quad \Big|\sum_{i\in S} x_i\Big| < \varepsilon$$

hold for all $r \geq k$ and all finite subsets S of $\mathbb{N}$ with $\min S \geq k$. Fix some $r > k$

such that for each $1 \le i \le k$ there exists $1 \le j \le r$ with $x_i = x_{\sigma_j}$, and note that

$$|x - y| \le \Big|x - \sum_{n=1}^{k} x_n\Big| + \Big|\sum_{n=1}^{k} x_n - \sum_{n=1}^{r} x_{\sigma_n}\Big| + \Big|\sum_{n=1}^{r} x_{\sigma_n} - y\Big| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

holds for all $\varepsilon > 0$, and so $x = y$. In other words, the series $\sum_{n=1}^{\infty} x_n$ is rearrangement invariant.

Problem 5.8. *A series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$, where $x_n > 0$ for each n, is called an* **alternating series**. *Assume that a sequence $\{x_n\}$ of strictly positive real numbers satisfies $x_n \downarrow 0$. Then establish the following:*

a. *The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$ converges in $\mathbb{R}$.*
b. *If $\sum_{n=1}^{\infty} x_n = \infty$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}x_n$ is not rearrangement invariant.*

Solution. (a) Let $s_n = \sum_{k=1}^{n}(-1)^{k-1}x_k$. We claim that

$$s_2 \le s_4 \le \cdots \le s_{2n-2} \le s_{2n} \le s_{2n-1} \le s_{2n-3} \le \cdots \le s_3 \le s_1$$

holds for each n. The proof is by induction. For $n = 1$, we have $s_2 = x_1 - x_2 < x_1 = s_1$. So, assume the inequalities to be true for some n. Then taking into account that $x_{2n} - x_{2n+1} \ge 0$ and $x_{2n+1} - x_{2n+2} \ge 0$, we see that

1. $s_{2n} \le s_{2n} + (x_{2n+1} - x_{2n+2}) = s_{2n+2} = s_{2(n+1)}$,
2. $s_{2(n+1)} = s_{2n+2} = s_{2n+1} - x_{2n+2} \le s_{2n+1} = s_{2(n+1)-1}$,
3. $s_{2(n+1)-1} = s_{2n+1} = s_{2n-1} - (x_{2n} - x_{2n+1}) \le s_{2n-1}$,

and our claim is established.

Now, if $s_{2n} \uparrow s$ and $s_{2n-1} \downarrow t$ hold in $\mathbb{R}$, then clearly $s \le t$. Moreover, from $s_{2n} - s_{2n-1} = -x_{2n} \to 0$, we obtain $s = t$. But then, this implies that $\{s_n\}$ converges to s in $\mathbb{R}$; see Exercise 4.3 of Section 4. Consequently, the alternating series converges and $\sum_{k=1}^{\infty}(-1)^{k-1}x_k = \lim s_n = s$.

(b) We must have either $\sum_{k=1}^{\infty} x_{2k-1} = \infty$ or $\sum_{k=1}^{\infty} x_{2k} = \infty$. Assume $\sum_{k=1}^{\infty} x_{2k-1} = \infty$; the other case can be treated in a similar manner.

Since $\sum_{k=1}^{\infty} x_{2k-1} = \infty$, there exist integers $0 = k_0 < k_1 < k_2 < \cdots$ such that $\left[\sum_{i=k_n+1}^{k_{n+1}} x_{2i-1}\right] - x_{2n} > 1$ holds for each $n = 0, 1, \ldots$. Consider the rearrangement $\{y_n\}$ of the sequence $\{(-1)^{n-1}x_n\}$ given by

$$x_1, x_3, \ldots, x_{2k_1-1}, -x_2, x_{2k_1+1}, \ldots, x_{2k_2-1}, -x_4, x_{2k_2+1}, \ldots,$$

and note that $\sum_{n=1}^{\infty} y_n = \infty$ holds.

Problem 5.9. *This problem describes the* **integral test** *for the convergence of series. Assume that $f: [1, \infty) \to [0, \infty)$ is a decreasing function. We define the sequences $\{\sigma_n\}$ and $\{\tau_n\}$ by*

$$\sigma_n = \sum_{k=1}^{n} f(k) \quad \text{and} \quad \tau_n = \int_1^n f(x)\,dx.$$

Establish the following:

a. $0 \le \sigma_n - \tau_n \le f(1)$ *for all n.*
b. *the sequence $\{\sigma_n - \tau_n\}$ is decreasing—and hence, convergent in* $\mathbb{R}$.
c. *Show that the series $\sum_{k=1}^{\infty} f(k)$ converges in* $\mathbb{R}$ *if and only if the improper Riemann integral $\int_1^{\infty} f(x)\,dx = \lim_{r\to\infty} \int_1^r f(x)\,dx$ exists in* $\mathbb{R}$.

Solution. Since f is decreasing, notice that for each $k \in \mathbb{N}$ we have $f(x) \ge f(k+1)$ and $f(x) \le f(k)$ for each $k \le x \le k+1$. So, integrating over $[k, k+1]$, we get:

$$\int_k^{k+1} f(x)\,dx \ge f(k+1), \text{ and} \tag{1}$$

$$\int_k^{k+1} f(x)\,dx \le f(k) \tag{2}$$

for each $k = 1, 2, \ldots$. (We remark that as a decreasing function, f is Riemann integrable over every closed subinterval of $[1, \infty)$; see Section 23 of the text.)

(a) Using (1), we see that

$$\begin{aligned}
\sigma_n &= f(1) + \sum_{k=2}^{n} f(k) = f(1) + \sum_{k=1}^{n-1} f(k+1) \\
&\le f(1) + \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx = f(1) + \int_1^n f(x)\,dx \\
&= f(1) + \tau_n.
\end{aligned}$$

This implies $\sigma_n - \tau_n \le f(1)$ for each n. On the other hand, using (2), we see that

$$\sigma_n \ge \sum_{k=1}^{n-1} f(k) \ge \sum_{k=1}^{n-1} \int_k^{k+1} f(x)\,dx = \int_1^n f(x)\,dx = \tau_n,$$

and so $\sigma_n - \tau_n \ge 0$ for each n.

(b) Using once more (1), we get

$$\begin{aligned}\sigma_{n+1} - \tau_{n+1} &= \sigma_n + f(n+1) - \tau_n - \int_n^{n+1} f(x)\,dx \\ &= \sigma_n - \tau_n - \left[\int_n^{n+1} f(x)\,dx - f(n+1)\right] \le \sigma_n - \tau_n.\end{aligned}$$

This shows that $\{\sigma_n - \tau_n\}$ is a decreasing sequence—and so $\lim(\sigma_n - \tau_n)$ exists in $\mathbf{R}$.

(c) Since $\{\sigma_n\}$ and $\{\tau_n\}$ are both increasing sequences of non-negative real numbers they both converge in $\mathbf{R}^*$, and clearly

$$\lim_{n\to\infty} \sigma_n = \sum_{k=1}^{\infty} f(k) \quad \text{and} \quad \lim_{n\to\infty} \tau_n = \int_1^{\infty} f(x)\,dx.$$

But from part (a), we have $\tau_n \le \sigma_n \le f(1) + \tau_n$ for each n, and therefore (by letting $n \to \infty$), we have

$$0 \le \int_1^{\infty} f(x)\,dx \le \sum_{k=1}^{\infty} f(k) \le f(1) + \int_1^{\infty} f(x)\,dx.$$

This inequality shows that $\sum_{k=1}^{\infty} f(k)$ converges if and only the improper Riemann integral $\int_1^{\infty} f(x)\,dx$ exists.

Problem 5.10. *Use the preceding problem to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ does not converge in $\mathbf{R}$ for $0 < p \le 1$ and converges in $\mathbf{R}$ for all $p > 1$. The following are problems related to the* **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$.

a. *Prove with (at least) three different ways that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.*
b. *If a computer starting at 12 midnight on December 31, 1939, adds one million terms of the harmonic series every second, what was the value (within an error of 1) of the sum at 12 midnight on December 31, 1997? (Assume that each year has 365 days.)*
c. *Show that* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \ln 2.$

Solution. Notice that

$$\int_1^{\infty} \frac{dx}{x^p} = \lim_{r\to\infty} \int_1^{r} \frac{dx}{x^p} = \begin{cases} \lim_{r\to\infty} \frac{r^{1-p}-1}{1-p} & \text{if } p \ne 1, \\ \lim_{r\to\infty} \ln r & \text{if } p = 1. \end{cases}$$

This limit is finite if $p > 1$ and infinity if $0 < p \le 1$.

(a) We let $\sigma_n = \sum_{k=1}^{n} \frac{1}{k}$. Here are four proofs of the divergence of the harmonic series.

1. Notice that $\sigma_{2n} - \sigma_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \geq n \cdot \frac{1}{2n} = \frac{1}{2}$ for each n. This shows that $\{\sigma_n\}$ is not a Cauchy sequence, and hence, divergent.
2. As shown at the beginning of the solution of the problem, $\int_1^\infty \frac{dx}{x} = \infty$, and so $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.
3. We claim that $\sigma_{2^n} \geq 1 + \frac{n}{2}$ for each n. (If this inequality is established, then clearly $\sum_{k=1}^{\infty} \frac{1}{k} = \lim \sigma_{2^n} = \infty$.) The proof of the inequality is by induction. For $n = 1$, we have $\sigma_{2^1} = \sigma_2 = 1 + \frac{1}{2}$. Now, if we assume the inequality true for some n, then

$$\begin{aligned}\sigma_{2^{n+1}} &= \sigma_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \\ &\geq 1 + \frac{n}{2} + 2^n \cdot \frac{1}{2 \cdot 2^n} = 1 + \frac{n+1}{2}.\end{aligned}$$

4. Note that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= \left[1 + \frac{1}{2} + \cdots + \frac{1}{9}\right] + \left[\frac{1}{10} + \cdots + \frac{1}{99}\right] + \left[\frac{1}{100} + \cdots + \frac{1}{999}\right] + \cdots \\ &\geq 9 \cdot \frac{1}{10} + 90 \cdot \frac{1}{100} + 900 \cdot \frac{1}{1000} + \cdots \\ &= \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \cdots = \infty.\end{aligned}$$

(b) From Problem 5.9, we know that the harmonic series is associated with the function $f(x) = \frac{1}{x}$ and that $0 \leq \sigma_n - \ln n \leq f(1) = 1$ for each n. So, $\ln n$ approximates σ_n within an error of one. If the computer started adding the terms of the harmonic series at 12 midnight on December 31, 1939, then up to 12 midnight on December 31, 1997, there are

$$\begin{aligned}&57(\text{years}) \times 365(\text{days}) \times 24(\text{hours}) \times 60(\text{minutes}) \times 60(\text{seconds}) \\ &= 1{,}797{,}552 \times 10^3 \text{ seconds.}\end{aligned}$$

So, if the computer adds $1,000,000 = 10^6$ terms per second of the harmonic series, the last number N added a second before midnight on December 31 of 1997 is $N = 1,797,552 \times 10^9$. Therefore,

$$\sum_{n=1}^{N} \frac{1}{n} = \sigma_N \approx \ln N = 35.12520213\ldots,$$

which shows that the harmonic series is a "very slow" divergent series.

(c) From Problem 5.8, we know that the alternating series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ is convergent in $\mathbb{R}$. Also, from Problem 5.9, we know that $\lim(\sigma_n - \ln n) = \gamma \in \mathbb{R}$. So, if we let $x_n = \gamma - (\sigma_n - \ln n)$, then $x_n \to 0$ and $\sigma_n = \gamma + \ln n + x_n$ for each n. Now, note that

$$\begin{aligned}
\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \\
&= \sigma_{2n} - \sigma_n = \left[\gamma + \ln(2n) + x_{2n}\right] - \left[\gamma + \ln n + x_n\right] \\
&= \ln 2 + x_{2n} - x_n,
\end{aligned}$$

for each n. This implies $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2$.

Problem 5.11 (Toeplitz). *Let $\{a_n\}$ be a sequence of positive real numbers (i.e., $a_n > 0$ for each n) and put $b_n = \sum_{i=1}^{n} a_i$. Assume that $b_n \uparrow \sum_{i=1}^{\infty} a_i = \infty$. If $\{x_n\}$ is a sequence of real numbers such that $x_n \to x$ in $\mathbb{R}$, then show that*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i x_i = x.$$

Solution. Let $\epsilon > 0$. Choose some k such that $|x_n - x| < \epsilon$ for each $n \geq k$. Put $M = \max\{|x_i - x|: i = 1, \ldots, k\}$, and then select some $\ell > k$ such that $\frac{Mb_k}{b_n} < \epsilon$ for all $n \geq \ell$. Now, notice that if $n \geq \ell$, then

$$\begin{aligned}
\left|\frac{1}{b_n}\sum_{i=1}^{n} a_i x_i - x\right| &= \left|\frac{1}{b_n}\sum_{i=1}^{n} a_i x_i - \frac{1}{b_n}\sum_{i=1}^{n} a_i x\right| \\
&\leq \frac{1}{b_n}\sum_{i=1}^{k} a_i |x_i - x| + \frac{1}{b_n}\sum_{i=k+1}^{n} a_i |x_i - x| \\
&\leq \frac{Mb_k}{b_n} + \epsilon < \epsilon + \epsilon = 2\epsilon,
\end{aligned}$$

and the conclusion follows. (Note that this problem is a substantial generalization of Problem 4.11.)

Problem 5.12 (Kronecker). *Assume that a sequence of positive numbers $\{b_n\}$ satisfies $0 < b_1 < b_2 < b_3 < \cdots$ and $b_n \uparrow \infty$. If a series $\sum_{n=1}^{\infty} x_n$ of real numbers*

converges in $\mathbb{R}$*, then show that*

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} b_i x_i = 0.$$

In particular, show that if $\{y_n\}$ *is a sequence of real numbers such that the series* $\sum_{n=1}^{\infty} \frac{y_n}{n}$ *converges in* $\mathbb{R}$*, then* $\frac{y_1+\cdots+y_n}{n} \to 0$.

Solution. Let $x = \sum_{n=1}^{\infty} x_n$. Put $b_0 = 0$, $s_0 = 0$, and $s_n = x_1 + \cdots + x_n$ for each $n \geq 1$. Now, notice that

$$\sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} b_i(s_i - s_{i-1}) = b_n s_n - \sum_{i=2}^{n} s_{i-1}(b_i - b_{i-1}).$$

Therefore, $\frac{1}{b_n}\sum_{i=1}^{n} b_i x_i = s_n - \frac{1}{b_n}\sum_{i=2}^{n} s_{i-1}(b_i - b_{i-1})$. Since $b_i - b_{i-1} > 0$ for each i and $\sum_{i=1}^{n}(b_i - b_{i-1}) = b_n \uparrow \infty$, it follows from the preceding problem that $\frac{1}{b_n}\sum_{i=2}^{n} s_{i-1}(b_i - b_{i-1}) = x$. Hence,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} b_i x_i = \lim_{n\to\infty}\left[s_n - \frac{1}{b_n}\sum_{i=2}^{n} s_{i-1}(b_i - b_{i-1})\right] = x - x = 0.$$

For the second part, notice that if $\sum_{n=1}^{\infty} \frac{y_n}{n}$ is convergent in $\mathbb{R}$, then let $b_n = n$ for each n and notice that (by the above)

$$\frac{1}{b_n}\sum_{i=1}^{n} b_i \frac{y_i}{i} = \frac{y_1 + \cdots + y_n}{n} \longrightarrow 0,$$

as desired.

6. METRIC SPACES

Problem 6.1. *For subsets A and B of a metric space* (X, d) *show that:*

a. $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$.
b. $A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}$.
c. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
d. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
e. *If B is open, then* $\overline{A} \cap B \subseteq \overline{A \cap B}$.

Solution. (a) From $(A \cap B)^{\circ} \subseteq A^{\circ}$ and $(A \cap B)^{\circ} \subseteq B^{\circ}$, it follows that $(A \cap B)^{\circ} \subseteq A^{\circ} \cap B^{\circ}$. On the other hand, since $A^{\circ} \cap B^{\circ} \subseteq A \cap B$ holds and $A^{\circ} \cap B^{\circ}$ is open, it easily follows that $A^{\circ} \cap B^{\circ} \subseteq (A \cap B)^{\circ}$.

(b) From $A \subseteq A \cup B$, it follows that $A^o \subseteq (A \cup B)^o$. Similarly, $B^o \subseteq (A \cup B)^o$, and the desired inclusion follows.

(c) From $S \subseteq \overline{S}$ and the fact that $\overline{S}$ is a closed set for any subset S, we see that

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \cup \overline{A \cup B} = \overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}.$$

(d) Since $A \cap B \subseteq \overline{A} \cap \overline{B}$, we have $\overline{A \cap B} \subseteq \overline{\overline{A} \cap \overline{B}} = \overline{A} \cap \overline{B}$.

(e) If $x \in \overline{A} \cap B$ and $r > 0$, then choose some $0 < \delta < r$ with $B(x, \delta) \subseteq B$, and note that

$$B(x, r) \cap (A \cap B) \supseteq B(x, \delta) \cap B \cap A = B(x, \delta) \cap A \neq \emptyset.$$

That is, $x \in \overline{A \cap B}$, and so $\overline{A} \cap B \subseteq \overline{A \cap B}$ holds.

Problem 6.2. *Show that in a Euclidean space $\mathbf{R}^n$ with the Euclidean distance, the closure of any open ball $B(a, r)$ is the closed ball $\{x \in \mathbf{R}^n: d(x, a) \leq r\}$. Give an example of a complete metric space for which the corresponding statement is false.*

Solution. Let $C(a, r) = \{x \in \mathbf{R}^k: d(a, x) \leq r\}$. Since $C(a, r)$ is a closed set, it follows that $\overline{B(a, r)} \subseteq C(a, r)$.

For the other inclusion, let $x \in C(a, r)$. For each n let $x_n = \frac{1}{n}a + \left(1 - \frac{1}{n}\right)x$. The inequalities

$$d(a, x_n) = \left(1 - \tfrac{1}{n}\right)d(a, x) \leq \left(1 - \tfrac{1}{n}\right)r < r \text{ and } d(x, x_n) = \tfrac{1}{n}d(a, x) \leq \tfrac{r}{n}$$

imply that $\{x_n\} \subseteq B(a, r)$ and $x_n \longrightarrow x$. Consequently, $x \in \overline{B(a, r)}$, and thus $C(a, r) \subseteq \overline{B(a, r)}$ also holds.

For a counterexample, consider $X = \{0, 1\}$ with the discrete distance, and note that X is a complete metric space. Also, observe that $\overline{B(0, 1)} = \{0\}$, while $C(0, 1) = \{0, 1\}$.

Problem 6.3. *If A is a nonempty subset of $\mathbf{R}$, then show that the set*

$$B = \left\{a \in \overline{A}\colon \textit{There exists some } \varepsilon > 0 \textit{ with } (a, a + \varepsilon) \cap A = \emptyset\right\}$$

is at-most countable.

Solution. For each $a \in B$ pick a rational number $r_a > a$ so that $(a, r_a) \cap A = \emptyset$. We claim that if $a, b \in B$ satisfy $a \neq b$, then $r_a \neq r_b$. Indeed, if $a < b$ and

$r_a = r_b$ hold, then—since $b \in (a, r_a)$—the open interval (a, r_a) is a neighborhood of $b \in \overline{A}$, and so $(a, r_a) \cap A \neq \emptyset$, contrary to the choice of r_a.

The above show that the mapping $a \longmapsto r_a$, from B into the set of rational numbers, is one-to-one. Consequently, the set B is at-most countable.

Problem 6.4. *Let $f:(X, d) \to (Y, \rho)$ be a function. Show that f is continuous if and only if $f^{-1}(B^{\mathrm{o}}) \subseteq \left[f^{-1}(B)\right]^{\mathrm{o}}$ for every subset B of Y.*

Solution. Assume f continuous, and $B \subseteq Y$. Since B^{o} is open, the set $f^{-1}(B^{\mathrm{o}})$ is likewise open. Thus, in view of $B^{\mathrm{o}} \subseteq B$, we have

$$f^{-1}(B^{\mathrm{o}}) = \left[f^{-1}(B^{\mathrm{o}})\right]^{\mathrm{o}} \subseteq \left[f^{-1}(B)\right]^{\mathrm{o}}.$$

In the opposite direction, assume that the condition is satisfied. If $B \subseteq Y$ is open (i.e., if $B = B^{\mathrm{o}}$ holds), then

$$\left[f^{-1}(B)\right]^{\mathrm{o}} \subseteq f^{-1}(B) = f^{-1}(B^{\mathrm{o}}) \subseteq \left[f^{-1}(B)\right]^{\mathrm{o}}$$

shows that $f^{-1}(B)$ is open. Therefore, f is continuous.

Problem 6.5. *Show that the boundary of a closed or open set in a metric space is nowhere dense. Is this statement true for an arbitrary subset?*

Solution. Since $\partial A = \partial A^{\mathrm{c}} = \overline{A} \cap \overline{A^{\mathrm{c}}}$ holds, we can assume that A is closed. Thus,

$$\begin{aligned}(\partial A)^{\mathrm{o}} &= \left(\overline{A} \cap \overline{A^{\mathrm{c}}}\right)^{\mathrm{o}} = \left(A \cap \overline{A^{\mathrm{c}}}\right)^{\mathrm{o}} = A^{\mathrm{o}} \cap \left(\overline{A^{\mathrm{c}}}\right)^{\mathrm{o}} \\ &\subseteq A^{\mathrm{o}} \cap \overline{A^{\mathrm{c}}} = A^{\mathrm{o}} \cap \left(A^{\mathrm{o}}\right)^{\mathrm{c}} = \emptyset.\end{aligned}$$

Since ∂A is closed, this shows that ∂A is nowhere dense.

An alternate proof goes as follows: If $x \in (\partial A)^{\mathrm{o}}$, then there exists some $r > 0$ such that $B(x, r) \subseteq \partial A = A \cap \overline{A^{\mathrm{c}}} \subseteq A$. This implies $B(x, r) \cap A^{\mathrm{c}} = \emptyset$, contrary to $x \in \overline{A^{\mathrm{c}}}$.

The boundary of an arbitrary set need not be nowhere dense. An example: Let $X = \mathbf{R}$ with the Euclidean distance, and let $A = Q$ (the set of rational numbers). Note that $\partial A = \mathbf{R}$.

Problem 6.6. *Show that the set of irrational numbers is not a countable union of closed subsets of $\mathbf{R}$.*

Solution. Let I denote the set of all irrational numbers, and let $\{r_1, r_2, \ldots\}$ be an enumeration of the rational numbers of $\mathbf{R}$.

Assume by way of contradiction that there exists a sequence of closed sets $\{A_n\}$ of $\mathbb{R}$ such that $I = \bigcup_{n=1}^{\infty} A_n$. Then

$$\mathbb{R} = \Big(\bigcup_{n=1}^{\infty} A_n\Big) \cup \Big(\bigcup_{n=1}^{\infty} \{r_n\}\Big),$$

and by the Baire Category Theorem (Theorem 6.18), we must have $(A_n)^o \neq \emptyset$ for some n. Thus, some A_n contains an interval. However, since $A_n \subseteq I$ holds and each interval contains rational numbers, this is impossible, and the conclusion follows.

Problem 6.7. *Let (X, d) be a metric space. Show that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of X, then $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.*

Solution. Use the inequality

$$\big|d(x_n, y_n) - d(x_m, y_m)\big| \leq d(x_n, x_m) + d(y_n, y_m).$$

(Also, see the discussion before Theorem 6.19.)

Problem 6.8. *Show that in a metric space a Cauchy sequence converges if and only if it has a convergent subsequence.*

Solution. Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, d). If $x_n \to x$ holds in X, then every subsequence of $\{x_n\}$ converges to x.

For the converse, assume that there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ such that $x_{k_n} \to x$ holds in X. Let $\epsilon > 0$. Choose n_0 such that $d\big(x_{k_n}, x\big) < \epsilon$ and $d(x_n, x_m) < \epsilon$ for $n, m > n_0$. Now, if $n > n_0$, then $k_n \geq n > n_0$, and so

$$d(x_n, x) \leq d\big(x_n, x_{k_n}\big) + d\big(x_{k_n}, x\big) < \epsilon + \epsilon = 2\epsilon.$$

This shows that, $\lim x_n = x$ holds in X.

Problem 6.9. *Prove that the closed interval* $[0, 1]$ *is an uncountable set:*

a. *by using Cantor's Theorem 6.14, and*
b. *by using Baire's Theorem 6.17.*

Solution. (a) Assume by way of contradiction that $[0, 1]$ is a countable set, say $[0, 1] = \{x_1, x_2, \ldots\}$. We consider $[0, 1]$ equipped with the usual distance $d(x, y) = |x - y|$ so that $[0, 1]$ is a complete metric space.

Subdivide $[0, 1]$ into three closed subintervals (as in the construction of the Cantor set) of equal length. Remove from $[0, 1]$ the middle open subinterval and consider the remaining two closed subintervals (here the subintervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$) and then select one of them, say I_1, such that $x_1 \notin I_1$. Next, repeat this process with I_1 in place of $[0, 1]$ and select a closed subinterval I_2 of I_1 of length equal to one-third of I_1 such that $x_2 \notin I_2$. Inductively, assume that we have chosen n closed intervals $I_1, \ldots, I_n$ such that:

1. $I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 \subseteq I_1$,
2. $x_k \notin I_k$ for $k = 1, \ldots, n$, and
3. the length of each I_k is $\frac{1}{3^k}$.

As above, there exists a closed subinterval I_{n+1} of I_n of length equal to one-third of I_n such that $x_{n+1} \notin I_{n+1}$.

Thus, there exists a sequence $\{I_n\}$ of closed subintevals of $[0, 1]$ such that $I_{n+1} \subseteq I_n$, $x_n \notin I_n$ and $d(I_n) = \frac{1}{3^n}$ for each n. By Theorem 6.14, we infer that $\bigcap_{n=1}^{\infty} I_n$ consists exactly of one point. But, since $x_n \notin I_n$ for each n, we see that $\bigcap_{n=1}^{\infty} I_n = \emptyset$, a contradiction. Hence, $[0, 1]$ must be uncountable.

(b) Again, assume by way of contradiction that $[0, 1] = \{x_1, x_2, \ldots\}$ and again we consider $[0, 1]$ as a complete metric space. If $A_n = \{x_n\}$, then each A_n is closed and has no interior points. However, Theorem 6.17 (or Theorem 6.18) applied to the equality $[0, 1] = \bigcup_{n=1}^{\infty} A_n$ implies that some A_n must have an interior point, which is impossible. This shows that $[0, 1]$ cannot be countable.

Problem 6.10. *Let $\{r_1, r_2, \ldots\}$ be an enumeration of all rational numbers in the interval $[0, 1]$ and for each $x \in [0, 1]$ let $A_x = \{n \in \mathbb{N}\colon r_n \le x\}$. Define the function $f\colon [0, 1] \to \mathbb{R}$ by the formula*

$$f(x) = \sum_{n \in A_x} \frac{1}{2^n}.$$

Show that f restricted to the set of irrational numbers of $[0, 1]$ is continuous.

Solution. Fix an irrational number $a \in [0, 1]$ and let $\varepsilon > 0$. Pick a natural number k such that $\sum_{n=k}^{\infty} \frac{1}{2^n} < \varepsilon$ and let

$$\delta = \min\{|a - r_1|, |a - r_2|, \ldots, |a - r_k|\} > 0.$$

We claim that if $x \in [0, 1]$ is an irrational number, then $|a - x| < \delta$ implies $|f(x) - f(a)| < \varepsilon$ (which tells us that f is continuous when restricted to the irrational numbers).

To see this, let $x \in [0, 1]$ be an arbitrary irrational number satisfying $|x - a| < \delta$. Let I_x denote the half-open subinterval of $[0, 1]$ which is open at left and closed

at right having endpoints a and x. If $B_x = \{n \in \mathbb{N}\colon r_n \in I_x\}$, then note that $B_x \subseteq \{k, k+1, k+2, \ldots\}$ (why?), and so

$$|f(x) - f(a)| = \sum_{n \in B_x} \frac{1}{2^n} \le \sum_{n=k}^{\infty} \frac{1}{2^n} < \varepsilon$$

holds, as claimed.

Problem 6.11. *This problem concerns connected metric spaces. A metric space (X, d) is said to be* **connected** *whenever $\emptyset$ and X are the only subsets of X that are simultaneously open and closed. A subset A of a metric space (X, d) is said to be* **connected** *whenever (A, d) is itself a connected metric space. Establish the following properties regarding connected metric spaces and connected sets.*

a. *A metric space (X, d) is connected if and only if every continuous function $f\colon X \to \{0, 1\}$ is constant, where the two point set $\{0, 1\}$ is considered to be a metric space under the discrete metric.*
b. *If in a metric space (X, d) we have $B \subseteq A \subseteq X$, then the set B is a connected subset of (A, d) if and only if B is a connected subset of (X, d).*
c. *If $f\colon (X, d) \to (Y, \rho)$ is a continuous function and A is a connected subset of X, then $f(A)$ is a connected subset of Y.*
d. *If $\{A_i\}_{i \in I}$ is a family of connected subsets of a metric space such that $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcup_{i \in I} A_i$ is likewise a connected set.*
e. *If A is a subset of a metric space and $a \in A$, then there exists a largest (with respect to inclusion) connected subset C_a of A that contains a. (The connected set C_a is called the* **component** *of a with respect to A.)*
f. *If a, b belong to a subset A of a metric space and C_a and C_b are the components of a and b in A, then either $C_a = C_b$ or else $C_a \cap C_b = \emptyset$. Hence, the identity $A = \bigcup_{a \in A} C_a$ shows that A can be written as a disjoint union of connected sets.*
g. *A nonempty subset of $\mathbb{R}$ with at least two elements is a connected set if and only if it is an interval. Use this and the conclusion of (f) to infer that every open subset of $\mathbb{R}$ can be written as an at-most countable union of disjoint open intervals.*

Solution. (a) Let (X, d) be a connected space and let $f\colon X \longrightarrow \{0, 1\}$ be a continuous function. Then the set $A = f^{-1}(0)$ is an open and closed subset of X. Since X is connected either $A = \emptyset$ (in which case $f(x) = 1$ holds for each $x \in X$) or $A = X$ (in which case $f(x) = 0$ holds for each $x \in X$).

For the converse, assume that every continuous function from X into $\{0, 1\}$ is constant and let A be a closed and open subset of X. Then the function

$f: X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A, \end{cases}$$

is continuous (why?). By our hypothesis, f must be a constant function, and this implies that either $A = \emptyset$ or $A = X$, i.e., X is a connected metric space.

(b) It follows immediately from (a).

(c) Assume that f and A satisfy the stated properties and consider the continuous functions $(A, d) \xrightarrow{f} (f(A), \rho) \xrightarrow{g} \{0, 1\}$. By (a), the continuous function $g \circ f$ must be a constant function and from this, we see that g is also a constant function. By (a), $(f(A), \rho)$ is a connected metric space.

(d) Assume the family $\{A_i\colon i \in I\}$ satisfies the stated properties. Put $A = \bigcup_{i \in I} A_i$ and let $f\colon (A, d) \longrightarrow \{0, 1\}$ be a continuous function. Then the function $f\colon (A_i, d) \longrightarrow \{0, 1\}$ is a continuous function and so f restricted to each A_i is constant. Since $\bigcap_{i \in I} A_i \neq \emptyset$, we see that f is constant on A, and so—by (a)—the set A is connected.

(e) Fix $a \in A$ and let

$$\mathcal{A} = \{B \subseteq A\colon\ B \text{ is connected and } a \in B\}.$$

Note that $\{a\} \in \mathcal{A}$ and that $\bigcap_{B \in \mathcal{A}} B \neq \emptyset$. By (d), the set $C_a = \bigcup_{B \in \mathcal{A}} B$ is a connected subset of A that satisfies the desired properties.

(f) If $C_a \cap C_b \neq \emptyset$, then by (d) we infer that $C_a \cup C_b$ is a connected set containing a. Hence, $C_b \subseteq C_a \cup C_b \subseteq C_a$. Similarly, $C_a \subseteq C_b$ and so $C_a = C_b$.

(g) Let A be a connected subset of $\mathbb{R}$ and let $a, b \in A$ satisfy $a < b$. If $a < x < b$ and $x \notin A$, then the set $A \cap (-\infty, x)$ is a proper and closed subset of A (why?), a contradiction. Thus, $(a, b) \subseteq A$ holds and this shows that A is an interval.

For the converse, assume that I is an interval of $\mathbb{R}$. Assume by way of contradiction that there exists an onto continuous function $f\colon I \longrightarrow \{0, 1\}$. Pick $a, b \in I$ such that $f(a) = 0$ and $f(b) = 1$; we can suppose that $a < b$ (and so $[a, b] \subseteq I$). Now, let

$$c_0 = \sup\{c \in [a, b)\colon\ f(c) = 0\}.$$

By the continuity of f, we see that $f(c_0) = 0$ and that $c_0 < b$. Then $f(x) = 1$ holds for all $c_0 < x < b$, and so (by the continuity of f again) $f(c_0) = 1$ must also hold, which is impossible. Therefore, every continuous function from I into $\{0, 1\}$ is constant, and so by (a) the interval I is a connected set.

Finally, note that if I is an open subset of $\mathbb{R}$, then by (f) we know that $I = \bigcup_{a \in I} C_a$, where each C_a is a connected set. It easily follows (how?) that each C_a is an open interval and that there are at-most countably many of them.

Problem 6.12. *Show that $\mathbb{R}^n$ with the Euclidean distance is a connected metric space. Use this conclusion to establish that, if the intersection of two open subsets of $\mathbb{R}^n$ is a proper closed set, then the two open sets must be disjoint.*

Solution. Let d denote the Euclidean distance of $\mathbb{R}^n$, i.e., let

$$d(\mathbf{x}, \mathbf{y}) = \Big[\sum_{i=1}^{n}(x_i - y_i)^2\Big]^{\frac{1}{2}}.$$

For each $\mathbf{x} \in \mathbb{R}^n$, let $L_{\mathbf{x}}$ denote the line segment joining $\mathbf{0}$ and $\mathbf{x}$, i.e., let $L_{\mathbf{x}} = \{t\mathbf{x}\colon 0 \le t \le 1\}$. We claim that $L_{\mathbf{x}}$ is a connected set.

To see this, note that the function $f\colon [0, 1] \longrightarrow L_{\mathbf{x}}$, defined by $f(t) = t\mathbf{x}$, satisfies $|f(t) - f(s)| \le d(\mathbf{x}, \mathbf{0})|s - t|$, and so f is (uniformly) continuous. From parts (g) and (c) of Problem 6.11, we see that $L_{\mathbf{x}}$ is a connected set.

Now, use part (d) of the preceding problem and the identity $\mathbb{R}^n = \bigcup_{\mathbf{x}\in\mathbb{R}^n} L_{\mathbf{x}}$ to infer that $\mathbb{R}^n$ is itself a connected metric space.

For the last part of the problem, let U and V be two open subsets of $\mathbb{R}^n$ such that $K = U \cap V$ is a closed set. Then K is both open and closed (and since K is a proper subset of $\mathbb{R}^n$) it must be the empty set.

Problem 6.13. *Let C be a nonempty closed subset of $\mathbb{R}$. Show that a function $f\colon C \to \mathbb{R}$ is continuous if and only if it can be extended to a continuous real-valued function on $\mathbb{R}$.*

Solution. Let C be a nonempty closed subset of $\mathbb{R}$ and let $f\colon C \to \mathbb{R}$ be a function. If f can be extended to a continuous real-valued function on $\mathbb{R}$, then $f\colon C \to \mathbb{R}$ is obviously continuous.

For the converse, assume that $f\colon C \to \mathbb{R}$ is a continuous function. Start by observing that the complement C^c of C is an open set and so (by part (g) of Problem 6.11) C^c can be written as an at-most countable union of pairwise disjoint open intervals; say $C^c = \bigcup_{i\in I}(a_i, b_i)$, where I is at-most countable. Since the open intervals $\{(a_i, b_i)\colon i \in I\}$ are pairwise disjoint, it follows that all the endpoints a_i and b_i belong to C. Hence, $f(a_i)$ and $f(b_i)$ are defined for each i. Now, extend the domain of f by defining the graph of the function f on the interval (a_i, b_i) to be the straight line segment joining the points $(a_i, f(a_i))$ and $(b_i, f(b_i))$. In other words, for each $a_i < x < b_i$ we let $f(x) = f(a_i) + \frac{f(b_i)-f(a_i)}{b_i-a_i}(x - a_i)$—in case $(a_i, b_i) = (-\infty, b_i)$ or $(a_i, b_i) = (a_i, \infty)$ let $f(x) = b_i$ or $f(x) = a_i$.

We claim that this extension of f to all of $\mathbb{R}$ is continuous. Clearly, f is continuous at every point of C^o and at every point of C^c (why?). We only need to verify that f is continuous at the boundary points of C. So, let $a \in \partial C$ and let $x_n \longrightarrow a$ with $\{x_n\} \subseteq C^c$—if $\{x_n\} \subseteq C$, then $f(x_n) \longrightarrow f(a)$ is trivially

true. Also, we shall assume that a is not one of the endpoints a_i or b_i. For each n pick (the unique) $i_n \in I$ with $a_{i_n} \le x_n \le b_{i_n}$. Note that in this case, we must have $\lim a_{i_n} = \lim b_{i_n} = a$ (why?). From

$$\begin{aligned} |f(x_n) - f(a)| &= \left|\left[f(a) + \tfrac{f(b_{i_n}) - f(a_{i_n})}{b_{i_n} - a_{i_n}}(x_n - a_{i_n})\right] - f(a)\right| \\ &= \left|\tfrac{f(b_{i_n}) - f(a_{i_n})}{a_{i_n} - b_{i_n}}(x_n - a_{i_n})\right| \le |f(b_{i_n}) - f(a_{i_n})| \\ &\longrightarrow |f(a) - f(a)| = 0, \end{aligned}$$

we see that $\lim f(x_n) = f(a)$. A similar conclusion holds true if a is one of the endpoints a_i or b_i. This shows that f is continuous at a, as claimed.

For an alternate proof see Problem 10.11.

Problem 6.14. *Show that a metric space is a Baire space if and only if the complement of every meager set is dense.*

Solution. Let X be a metric space. Assume first that X is a Baire space and let A be a meager set. Pick a sequence $\{A_n\}$ of nowhere dense sets such that $A = \bigcup_{n=1}^{\infty} A_n$. To show that A^c is dense, it suffices to show that $V \cap A^c \neq \emptyset$ for each nonempty open set V. To see this, let V be a nonempty open set, and assume by way of contradiction that $V \cap A^c = \emptyset$. This implies $V \subseteq A$, and so

$$V = \bigcup_{n=1}^{\infty} V \cap A_n.$$

Hence, V is a nonempty open meager set, a contradiction. Hence, A^c is a dense set.

For the converse, assume that the complement of every meager set is dense, and let V be an open meager set. Then V^c is dense. So, if V is nonempty, then $V \cap V^c \neq \emptyset$, which is impossible. Thus, the empty set is the only open meager set, and hence, X is a Baire space.

Problem 6.15. *A subset of a metric space is called* **co-meager** *if its complement is a meager set. For a subset A of a Baire space show that:*

a. *A is co-meager if and only if it contains a dense G_δ-set.*
b. *A is meager if and only if it is contained in an F_σ-set whose complement is dense.*

Solution. Notice that if A is a nowhere dense set in a metric space X, then from Lemma 6.8 we see that

$$\emptyset = (\overline{A})^{\circ} = (\overline{A})^{c-c} = ([(\overline{A})^c]^-)^c.$$

This implies that a subset A is nowhere dense if and only if the open set $(\overline{A})^c$ is dense.

Now, assume that X is a Baire space and let A be a subset of X.
(a) Suppose first that A is a co-meager set. Then there exists a sequence $\{A_n\}$ of nowhere dense sets such that $A = \left(\bigcup_{n=1}^{\infty} A_n\right)^c$. This implies

$$A^c = \bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} \overline{A_n}. \qquad (\star)$$

By the above discussion, each set $(\overline{A_n})^c$ is an open dense set, and since X is a Baire space, the G_δ-set $E = \bigcap_{n=1}^{\infty} (\overline{A_n})^c$ is also dense (see Theorem 6.16). Now, a glance at $(\star)$ shows that $E \subseteq A$.

For the converse, assume that A contains a dense G_δ-set B, i.e., $B \subseteq A$. So, there exists a sequence $\{V_n\}$ of open sets such that $B = \bigcap_{n=1}^{\infty} V_n$. From $B \subseteq V_n$, we see that each V_n is also dense. This implies

$$[(V_n)^c]^o = [(V_n)^c]^{c-c} = (\overline{V_n})^c = X^c = \emptyset,$$

and so each $(V_n)^c$ is nowhere dense closed set. Now, use the inclusion

$$A^c \subseteq B^c = \bigcup_{n=1}^{\infty} (V_n)^c$$

to conclude that A^c is a meager set, i.e., A is a co-meager set.
(b) Assume first that A is a meager set, i.e., A^c is a co-meager set. By part (a), there exists a dense G_δ-set E such that $E \subseteq A^c$. This implies $A \subseteq E^c$, where now E is an F_σ-set whose complement $(E^c)^c = E$ is dense.

For the converse, assume that $A \subseteq F$ holds, where F is an F_σ-set with dense complement. It follows that $F^c \subseteq A^c$, where now F^c is a dense G_δ-set. By part (a), A^c is co-meager set, which means that A is a meager set.

7. COMPACTNESS IN METRIC SPACES

Problem 7.1. *Let $f:(X,d) \to (Y,\rho)$ be a function. Show that f is continuous if and only if f restricted to the compact subsets of X is continuous.*

Solution. Assume that f is continuous on every compact set. Let $x_n \to x$. Then the set $A = \{x_1, x_2, \ldots\} \cup \{x\}$ is compact (note that every open cover of A can be reduced to a finite cover), and $x_n \to x$ holds in A. Since f restricted to A is continuous, $\lim f(x_n) = f(x)$ holds, which shows that f is continuous.

Problem 7.2. *A metric space is said to be* **separable** *if it contains a countable subset that is dense in the space. Show that every compact space (X, d) is separable.*

Solution. For each n choose a finite subset F_n of X such that $X = \bigcup_{x \in F_n} B(x, \frac{1}{n})$. Let $F = \bigcup_{n=1}^{\infty} F_n$, and note that F is at-most countable.

Now, let $x \in X$ and $r > 0$. Pick some n with $\frac{1}{n} < r$. Then there exists some $y \in F_n$ with $d(x, y) < \frac{1}{n} < r$. Thus, $B(x, r) \cap F \neq \emptyset$, and so F is dense in X.

Problem 7.3. *Show that if (X, d) is a separable metric space (see the preceding exercise for the definition), then* card $X \leq \mathfrak{c}$.

Solution. Let $\{x_1, x_2, \ldots\}$ be a countable dense subset of X. Consider the collection of open balls $\{B(x_i, \frac{1}{j})\colon\ i, j = 1, 2, \ldots\}$. Clearly, this collection is countable; let $\{B_1, B_2, \ldots\}$ be one of its enumerations. Now, for each $x \in X$ define the set $S_x = \{n \in \mathbb{N}\colon\ x \in B_n\}$. Thus, a mapping $x \longmapsto S_x$ from X into $\mathcal{P}(\mathbb{N})$ has been established that is clearly one-to-one. Consequently, $\operatorname{card} X \leq \operatorname{card} \mathcal{P}(\mathbb{N}) = \mathfrak{c}$. (See also Problem 5.6.)

Problem 7.4. *Let $(X_1, d_1), \ldots, (X_n, d_n)$ be arbitrary metric spaces, and let $X = X_1 \times \cdots \times X_n$. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, define*

$$D_1(x, y) = \sum_{m=1}^{n} d_m(x_m, y_m) \quad \textit{and} \quad D_2(x, y) = \Big(\sum_{m=1}^{n} [d_m(x_m, y_m)]^2\Big)^{\frac{1}{2}}.$$

a. *Show that D_1 and D_2 are distances on X.*
b. *Show that D_1 is equivalent to D_2.*
c. *Show that (X, D_1) is complete if and only if each (X_i, d_i) is complete.*
d. *Show that (X, D_1) is compact if and only if each (X_i, d_i) is compact.*

Solution. (a) Routine.

(b) Use the inequalities

$$\tfrac{1}{n} D_1(x, y) \leq D_2(x, y) \leq n D_1(x, y).$$

(c) Assume that each X_m $(m = 1, \ldots, n)$ is a complete metric space. Let $\{x_k\}$ be a D_1-Cauchy sequence of X, where $x_k = (x_1^k, \ldots, x_n^k)$. Clearly, each $\{x_m^k\}$ is a Cauchy sequence of X_m, and thus there exists $x_m \in X_m$ such that $\lim_{k\to\infty} d_m(x_m^k, x_m) = 0$. Hence, if $x = (x_1, \ldots, x_n) \in X$, then we have $\lim_{k\to\infty} D_1(x_k, x) = 0$, so that the metric space X is D_1-complete.

Now, let X be D_1-complete. Fix an element $(y_1, \ldots, y_n) \in X$. Let $\{x_m^k\}$ be a Cauchy sequence of X_m. If $x_k \in X$ is the element whose j$^{\text{th}}$ component equals y_j for $j \neq m$ and equals x_m^k if $j = m$, then $\{x_k\}$ is a Cauchy sequence of X. If $x \in X$ is its limit, then it is easy to see that $\lim_{k\to\infty} d_m(x_m^k, x_m) = 0$, so that each X_m is complete.

(d) Assume first that each X_m is compact. Then following the proof of the second part of Theorem 7.4, we can see that every sequence of X has a convergent subsequence, and so X must be a compact metric space.

On the other hand, if X is a compact metric space, then the function $f_m \colon X \longrightarrow X_m$, defined by $f_m(x_1, \ldots, x_n) = x_m$, is continuous and onto for each $1 \leq m \leq n$. Hence, by Theorem 7.5, each $X_m = f_m(X)$ is compact.

Problem 7.5. *Let $\{(X_n, d_n)\}$ be a sequence of metric spaces, and let $X = \prod_{n=1}^{\infty} X_n$. For each $x = \{x_n\}$ and $y = \{y_n\}$ in X, define*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

a. *Show that d is a distance on X.*
b. *Show that (X, d) is a complete metric space if and only if each (X_n, d_n) is complete.*
c. *Show that (X, d) is a compact metric space if and only if each (X_n, d_n) is compact.*

Solution. (a) Note first that if d is a distance on a set X, then $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is likewise a distance on X, which is equivalent to d. From this observation it easily follows that

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

is a distance on $X = \prod_{n=1}^{\infty} X_n$.

(b) Let $\{x^k\}$ be a sequence of X, where $x^k = (x_1^k, x_2^k, \ldots)$. The proof follows from the following two properties (whose verifications are straightforward).

1. $x^k \longrightarrow x$ holds in X if and only if $x_i^k \longrightarrow x_i$ holds in X_i for each i; and
2. $\{x^k\}$ is a Cauchy sequence in X if and only if $\{x_i^k\}$ is a Cauchy sequence in X_i for each i.

(c) Assume that (X, d) is a compact metric space. Then the function $f_i \colon X \longrightarrow X_i$, defined by $f(x) = x_i$ for each $x = (x_1, x_2, \ldots) \in X$, is continuous and onto. By Theorem 7.5, each X_i is a compact metric space.

For the converse, assume that each X_i is a compact metric space. By (b), X is a complete metric space, and so by Theorem 7.8 it suffices to show that X is totally bounded. To this end, let $\varepsilon > 0$. Choose n such that $2^{-n} < \varepsilon$, and note that

$$\rho_n(x, y) = \sum_{i=1}^{n} \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

defines a distance on $\prod_{i=1}^{n} X_i$. It should be clear that ρ_n is equivalent to the distances of the preceding problem, and $\left(\prod_{i=1}^{n} X_i, \rho_n\right)$ is a compact metric space. Choose a finite subset F of $\prod_{i=1}^{n} X_i$ such that the ρ_n-balls with centers at the points of F and radii ε cover $\prod_{i=1}^{n} X_i$. Next, extend each $x \in F$ to an element of X (i.e., add to each $x \in F$ arbitrary components $x_{n+1}, x_{n+2}, \ldots$). Now, if $y = (y_1, y_2, \ldots) \in X$, then pick some $x \in F$ with $\rho_n(x, y) < \varepsilon$, and note that

$$d(x, y) = \rho_n(x, y) + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} < \varepsilon + 2^{-n} < \varepsilon + \varepsilon = 2\varepsilon.$$

Thus, $X = \bigcup_{x \in F} B(x, 2\varepsilon)$ holds, and therefore, X is totally bounded.

Problem 7.6. *A family of set $\mathcal{F}$ is said to have the* **finite intersection property** *if every finite intersection of sets of $\mathcal{F}$ is nonempty. Show that a metric space is compact if and only if every family of closed sets with the finite intersection property has a nonempty intersection.*

Solution. Let X be compact, and let $\{A_i \colon i \in I\}$ be a family of closed sets with the finite intersection property. If $\bigcap_{i \in I} A_i = \emptyset$, then $X = \bigcup_{i \in I} A_i^c$ holds, and by the compactness of X, there exist $i_1, \ldots, i_n \in I$ such that $X = \bigcup_{j=1}^{n} A_{i_j}^c$. Thus, $\bigcap_{j=1}^{n} A_{i_j} = \emptyset$, contrary to our hypothesis. Hence, $\bigcap_{i \in I} A_i \neq \emptyset$.

For the converse, assume that every family of closed sets with the finite intersection property has a nonempty intersection. Let $X = \bigcup_{i \in I} V_i$ be an open cover. Then $\bigcap_{i \in I} V_i^c = \emptyset$, and since $\{V_i^c \colon i \in I\}$ is a family of closed sets, our hypothesis guarantees the existence of a finite number of indices $i_1, \ldots, i_n$ such that $\bigcap_{j=1}^{n} V_{i_j}^c = \emptyset$. Thus, $X = \bigcup_{j=1}^{n} V_{i_j}$ holds, so that X is a compact metric space.

Problem 7.7. *Let $f \colon X \to X$ be a function from a set X into itself. A point $a \in X$ is called a* **fixed point** *for f if $f(a) = a$.*

Assume that (X, d) is a compact metric space and $f \colon X \to X$ satisfies the inequality $d(f(x), f(y)) < d(x, y)$ for $x \neq y$. Show that f has a unique fixed point.

Solution. Note first that f has at most one fixed point. Indeed, if $f(x) = x$ and $f(y) = y$ hold with $x \neq y$, then

$$d(x, y) = d\big(f(x), f(y)\big) < d(x, y)$$

must hold, which is absurd.

Now, define the function $g\colon X \to \mathbb{R}$ by $g(x) = d\big(x, f(x)\big)$. From the inequality

$$\big|g(x) - g(y)\big| \leq d\big(f(x), f(y)\big) + d(x, y) \leq 2d(x, y)$$

(see the discussion preceding Theorem 6.19), it follows that g is continuous. Since X is compact, g attains its minimum at some point $a \in X$. If $f(a) \neq a$, then the inequality

$$g\big(f(a)\big) = d\big(f(a), f(f(a))\big) < d\big(a, f(a)\big) = g(a)$$

shows that g does not attain a minimum at a. Thus, $f(a) = a$ must hold, and so a is a (unique) fixed point for f.

Problem 7.8. *Let (X, d) be a metric space. A function $f\colon X \to X$ is called a* **contraction** *if there exists some $0 < \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all $x, y \in X$; α is called a* **contraction constant**.

Show that every contraction f on a complete metric space (X, d) has a unique fixed point; that is, show that there exists a unique point $x \in X$ such that $f(x) = x$.

Solution. Note first that if $f(x) = x$ and $f(y) = y$ hold, then the inequality $d(x, y) = d\big(f(x), f(y)\big) \leq \alpha d(x, y)$ easily implies that $d(x, y) = 0$, and so $x = y$. That is, f has at-most one fixed point.

To see that f has a fixed point, choose some $a \in X$, and then define the sequence $\{x_n\}$ inductively by

$$x_1 = a \quad \text{and} \quad x_{n+1} = f(x_n) \text{ for } n = 1, 2, \ldots .$$

From our condition, it follows that

$$d(x_{n+1}, x_n) = d\big(f(x_n), f(x_{n-1})\big) \leq \alpha d(x_n, x_{n-1})$$

holds for $n = 2, 3, \ldots$. Thus, as in Problem 4.15, we can show that $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ is a convergent sequence. Let $x = \lim x_n$. Now, by observing that f is (uniformly) continuous, we obtain that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x),$$

and so x is a (unique) fixed point for f.

Problem 7.9. *A property of a metric space is called a* **topological property** *if it is preserved in a homeomorphic metric space.*

a. *Show that compactness is a topological property.*
b. *Show that completeness, boundedness, and total boundedness are not topological properties.*

Solution. (a) It follows from Theorem 7.5.

(b) Consider $(0, 1]$ and $[1, \infty)$ as metric spaces under the usual Euclidean distance $d(x, y) = |x - y|$. Clearly, $(0, 1]$ is not complete but it is bounded and totally bounded. Also, $[1, \infty)$ is complete (because it is a closed subset of $\mathbf{R}$), but is neither bounded nor totally bounded. On the other hand, $f: (0, 1] \to [1, \infty)$, defined by $f(x) = \frac{1}{x}$, is a homeomorphism, and the claims in (b) follow.

Problem 7.10. *Let (X, d) be a metric space. Define the distance of two nonempty subsets A and B of X by*

$$d(A, B) = \inf\{d(x, y)\colon\ x \in A \text{ and } y \in B\}.$$

a. *Give an example of two closed sets A and B of some metric space with $A \cap B = \emptyset$ and such that $d(A, B) = 0$.*
b. *If $A \cap B = \emptyset$, A is closed, and B is compact (and, of course, both are nonempty), then show that $d(A, B) > 0$.*

Solution. (a) Let $X = \mathbf{R}^2$ with the Euclidean distance, and consider the closed subsets of X

$$A = \{(x, \tfrac{1}{x})\colon\ x \geq 1\} \quad \text{and} \quad B = \{(x, 0)\colon\ x \geq 1\}.$$

Note that $A \cap B = \emptyset$, while $d(A, B) = 0$.

(b) Let A and B be as stated in the problem. If $d(A, B) = 0$, then pick two sequences $\{x_n\} \subseteq A$ and $\{y_n\} \subseteq B$ with $d(x_n, y_n) \longrightarrow 0$. Since B is compact, by passing to a subsequence (if necessary), we can assume that $y_n \longrightarrow y$ holds for some $y \in B$. The inequality

$$d(x_n, y) \leq d(x_n, y_n) + d(y_n, y)$$

shows that $d(x_n, y) \longrightarrow 0$. Since A is closed, $y \in A$, and hence $A \cap B \neq \emptyset$, contrary to our hypothesis. Therefore, $d(A, B) > 0$ must hold.

Problem 7.11. *Let (X, d) be a compact metric space and $f: X \to X$ an isometry; that is, $d(f(x), f(y)) = d(x, y)$ holds for all $x, y \in X$. Then show that f is onto. Does the conclusion remain true if X is not assumed to be compact?*

Solution. Let $y \in X$. Define the sequence $\{x_n\}$ of $f(X)$ by

$$x_1 = f(y) \quad \text{and} \quad x_{n+1} = f(x_n) \text{ for } n = 1, 2, \ldots .$$

Note that $d(x_n, x_{n+p}) = d(y, x_p)$ holds for all n and all p. Since $f(X)$ is compact, $\{x_n\}$ must have a limit point in $f(X)$. Let a be a limit point of $\{x_n\}$.

Now, let $\varepsilon > 0$. Pick $n > 1$ and p such that $d(x_n, a) < \varepsilon$ and $d(x_{n+p}, a) < \varepsilon$. Then

$$d\big(y, f(X)\big) \le d(y, x_p) = d(x_n, x_{n+p}) \le d(x_n, a) + d(x_{n+p}, a) < 2\varepsilon$$

holds for all $\varepsilon > 0$, and so $d\big(y, f(X)\big) = 0$. Thus, $y \in \overline{f(X)} = f(X)$, so that $f(X) = X$ holds.

If X is not supposed to be compact, then the conclusion is no longer true. A counterexample: Take $X = \mathbb{N}$ with $d(n, m) = |n - m|$ and consider the function $f: \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$.

Problem 7.12. *Show that a metric space X is compact if and only if every continuous real-valued function on X attains its maximum value.*

Solution. Let (X, d) be a metric space. Assume that X is compact and that $f: X \to \mathbb{R}$ is a continuous function. By Theorem 7.5, we know that $f(X)$ is a compact subset of $\mathbb{R}$, and so (by Theorem 7.4) $f(X)$ is closed and bounded. The maximum of $f(X)$ is the maximum value of f on X.

For the converse, assume that every continuous real-valued function on X attains a maximum value. Clearly, every continuous real-valued function on X attains also a minimum value.

We shall establish first that X is a complete metric space. Let $(\hat{X}, \hat{d})$ denote the completion of (X, d) and let $\hat{x} \in \hat{X}$. The function $f: X \longrightarrow \mathbb{R}$, defined by $f(x) = \hat{d}(\hat{x}, x)$, satisfies $\inf\{f(x)\colon x \in X\} = 0$. So, there exists some $x_0 \in X$ satisfying $f(x_0) = \hat{d}(\hat{x}, x_0) = 0$. It follows that $\hat{x} = x_0 \in X$ and so $\hat{X} = X$. This means that X is a complete metric space.

Next, we shall show that X is totally bounded. To establish this, assume by way of contradiction that X is not totally bounded. Then an easy inductive argument shows that there exist some $\varepsilon > 0$ and a sequence $\{x_n\}$ of X such that $d(x_n, x_m) \ge 3\varepsilon$ holds for $n \ne m$. For each n consider the nonempty closed set

$$C_n = [B(x_n, \varepsilon)]^c = \{x \in X\colon d(x, x_n) \ge \varepsilon\},$$

and then define the function $f_n: X \to \mathbb{R}$ by

$$f_n(x) = d(x, C_n) = \inf\{d(x, y)\colon y \in C_n\}.$$

So, f_n is a bounded function, $f_n(x) = 0$ holds for each $x \in C_n$ and $f_n(x_n) > 0$. Multiplying by a constant c_n, we can assume that $\sup\{f_n(x)\colon x \in X\} > n$ holds for each n. Now, define the function $f\colon X \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} f_n(x), & \text{if } x \in B(x_n, \varepsilon) \\ 0, & \text{if } x \notin \bigcup_{n=1}^{\infty} B(x_n, \varepsilon), \end{cases}$$

and we claim that f is a continuous function. Clearly, f is continuous at the points of the balls $B(x_n, \varepsilon)$. If $x_0 \notin \bigcup_{n=1}^{\infty} B(x_n, \varepsilon)$, note that $B(x_0, \frac{\varepsilon}{2}) \cap B(x_n, \varepsilon) \neq \emptyset$ holds for at-most one n (why?). If $B(x_0, \frac{\varepsilon}{2}) \cap B(x_n, \varepsilon) = \emptyset$ for each n, then $f(x) = 0$ for each x in $B(x_0, \frac{\varepsilon}{2})$, and so f is continuous at x_0. Thus, we can assume that $B(x_0, \frac{\varepsilon}{2}) \cap B(x_n, \varepsilon) \neq \emptyset$ for some n. We distinguish two cases.

CASE I: $d(x_0, x_n) > \varepsilon$.

In this case, there exists some $0 < r < \frac{\varepsilon}{2}$ such that $B(x_0, r) \cap B(x_n, \varepsilon) = \emptyset$. Clearly, $f(x) = 0$ holds for each $x \in B(x_0, r)$, and from this we see that f is continuous at x_0.

CASE II: $d(x_0, x_n) = \varepsilon$.

Let $\{z_k\}$ be a sequence of X satisfying $z_k \longrightarrow x_0$; we can assume that z_k belongs to $B(x_0, \frac{\varepsilon}{2})$ for each k. Note that if $z_k \notin B(x_n, \varepsilon)$, then $f(z_k) = 0$. On the other hand, if $z_k \in B(x_n, \varepsilon)$, then

$$0 \le f(z_k) = c_n d(z_k, C_n) \le c_n d(z_k, x_0).$$

Thus, $0 \le f(z_k) \le c_n d(z_k, x_0)$ holds for each k. In view of

$$\lim_{k\to\infty} d(z_k, x_0) = 0,$$

we see that $\lim f(z_k) = 0 = f(x_0)$ and so f is continuous at x_0 in this case too.

To contradict our hypothesis, note that f does not attain a maximum value. Thus, X must also be totally bounded. By Theorem 7.8, we see that X is a compact metric
space.

Problem 7.13. *This exercise presents a converse of Theorem 7.7. Assume that (X, d) is a metric space such that every real-valued continuous function on X is uniformly continuous.*

a. *Show that X is a complete metric space.*
b. *Give an example of a noncompact metric space with the above property.*
c. *If X has a finite number of isolated points (an element $a \in X$ is said to be an* **isolated point** *whenever there exists some positive $r > 0$ such that $B(a, r) \cap (X \setminus \{a\}) = \emptyset$), then show that X is a compact metric space.*

Solution. Let (X, d) be a metric space such that every continuous real-valued function on X is uniformly continuous.

(a) If $\hat{x} \in \hat{X}$ (the completion of X) is an element that does not belong to X, then the function $f: X \to \mathbb{R}$ defined by $f(x) = \frac{1}{d(\hat{x}, x)}$, $x \in X$, is a continuous real-valued function on X that fails to be uniformly continuous (why?), a contradiction. Hence, $\hat{X} = X$ holds, which means that X is a complete metric space.

(b) Let $X = \{1, 2, \dots\}$ equipped with the discrete distance d. Then every set is open and so every real-valued function f on X is continuous. Since $d(x, y) < 1$ implies $x = y$ (and so $f(x) - f(y) = 0$), we see that every real-valued function on X is uniformly continuous. Now, note that X is not a compact metric space.

(c) In view of (a), we need to establish that X is totally bounded. To this end, assume that X is not totally bounded. Then, there exist some $\varepsilon > 0$ and a sequence of elements $\{x_n\}$ of X such that $d(x_n, x_m) > 3\varepsilon$ for $n \neq m$. From our hypothesis, we can suppose that each x_n is an accumulation point of X. For each n pick an element y_n such that $0 < d(x_n, y_n) < \frac{\varepsilon}{n}$ and let $r_n = d(x_n, y_n)$. Put

$$C_n = \{x \in X: d(x, x_n) \geq r_n\}$$

and define the functions f_n and f as in the solution of Problem 7.12 (the open ball $B(x_n, \varepsilon)$ is now replaced by $B(x_n, r_n)$). Then f is a continuous function and satisfies $f(y_n) = 0$ for each n. Pick $z_n \in B(x_n, r_n)$ such that $f(x_n) > n$, and note that

$$\left|f(y_n) - f(z_n)\right| > n \quad \text{and} \quad \lim_{n\to\infty} d(y_n, z_n) = 0.$$

This shows that the continuous function f is not uniformly continuous, contrary to our hypothesis. Hence, X is totally bounded, as desired.

Problem 7.14. *Consider a function $f: (X, d) \to (Y, \rho)$ between two metric spaces. The graph G of f is the subset of $X \times Y$ defined by*

$$G = \{(x, y) \in X \times Y: y = f(x)\}.$$

If (Y, ρ) is a compact metric space, then show that f is continuous if and only if G is a closed subset of $X \times Y$, where $X \times Y$ is considered to be a metric space under the distance $D((x, y), (u, v)) = d(x, u) + \rho(y, v)$; see Problem 7.4. Does the result hold true if (Y, ρ) is not assumed to be compact?

Solution. Observe that an arbitrary sequence $\{(x_n, y_n)\}$ of $X \times Y$ satisfies $(x_n, y_n) \to (x, y)$ in $X \times Y$ if and only if $x_n \to x$ and $y_n \to y$ both hold.

Assume (Y, ρ) compact and G closed. If f is not continuous, then there exists a sequence $\{x_n\}$ of X and some $\varepsilon > 0$ such that $x_n \to x$ and $\rho\big(f(x_n), f(x)\big) \geq \varepsilon$ for all n (why?). Since (Y, ρ) is compact, by passing to a subsequence, we can assume that $f(y_n) \to y$ holds in Y. Now, observe that $\big(x_n, f(x_n)\big) \in G$ holds for each n and $\big(x_n, f(x_n)\big) \to (x, y)$ holds in $X \times Y$. Since G is closed, it follows that $(x, y) \in G$ and so $y = f(x)$. This implies

$$\rho\big(f(x_n), f(x)\big) \to \rho\big(f(x), f(x)\big) = 0,$$

which contradicts $\rho\big(f(x_n), f(x)\big) \geq \varepsilon$ for all n. Hence, f is a continuous function.

If (Y, ρ) is not compact, then a function with closed graph need not be continuous. For an example, consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0 . \end{cases}$$

Problem 7.15. *A cover $\{V_i\}_{i \in I}$ of a set X is said to be a* **pointwise finite cover** *whenever each $x \in X$ belongs at-most to a finite number of the V_i.*

Show that a metric space is compact if and only if every pointwise finite open cover of the space contains a finite subcover.

Solution. Clearly, if X is compact, then every pointwise finite open cover of X contains a finite subcover. For the converse, assume that every pointwise finite open cover of X contains a finite subcover. To establish that the metric space X is compact, it suffices to show that every sequence in X contains a convergence subsequence.

Let $\{x_n\}$ be a sequence in X. We can suppose (why?) that the sequence consists of distinct elements. Suppose by way of contradiction that $\{x_n\}$ has no convergence subsequence. Then x_1 is not in the closure of the set $\{x_n\colon n \neq 1\}$ and thus, there exists an open ball $V_1 = B(x_1, \delta_1)$ about x_1 with radius $0 < \delta_1 < 1$ and satisfying $x_n \notin V_1$ for all $n \neq 1$. Also, x_2 is not in the closure of the set $\{x_n\colon n \neq 2\}$ and thus, there exists an open ball $V_2 = B(x_2, \delta_2)$ about x_2 with radius $0 < \delta_2 < \frac{1}{2}$ and such that $x_n \notin V_2$ for all $n \neq 2$. Proceeding inductively, we see that for each k there exists an open ball $V_k = B(x_k, \delta_k)$ with radius $0 < \delta_k < \frac{1}{2^k}$ satisfying $x_n \notin V_k$ for all $n \neq k$.

Since the set $F = \{x_1, x_2, \ldots\}$ contains no convergent subsequences, the set F must contain all of its closure points. Thus, F is a closed set, and hence, the set $G = X \setminus F$ is an open set. Then, the collection $C = \{G, V_1, V_2, \ldots\}$ is an open cover of X. In fact, the collection C is a pointwise finite open cover of X because if a point x belongs to an infinite number of sets in C, then x belongs to an infinite number of the sets V_n. However, this would imply that a subsequence

of $\{x_n\}$ converges to the point x. Since the sequence $\{x_n\}$ contains no convergent subsequences, we infer that C is a pointwise finite open cover.

Therefore, C contains a finite subcover of X, say $V_1, \ldots, V_m, G$. Since G does not intersect $\{x_1, x_2, \ldots\}$, it follows that $\{x_1, x_2, \ldots\} \subseteq \bigcup_{i=1}^{m} V_i$. However, this contradicts the fact $x_n \notin V_k$ for $n \neq k$. Conclusion: The sequence $\{x_n\}$ must have a convergent subsequence—and hence, the metric space X is compact.

CHAPTER 2

TOPOLOGY AND CONTINUITY

8. TOPOLOGICAL SPACES

Problem 8.1. *For any subset A of a topological space show the following:*

a. $A^o = (\overline{A^c})^c$.
b. $\partial A = \overline{A} \setminus A^o$.
c. $(A \setminus A^o)^o = \emptyset$.

Solution. (a) Note that

$$\begin{aligned} x \in A^o &\iff \text{there exists a neighborhood } V \text{ of } x \text{ with } V \subseteq A \\ &\iff \text{there exists a neighborhood } V \text{ of } x \text{ with } V \cap A^c = \emptyset \\ &\iff x \notin \overline{A^c} \iff x \in (\overline{A^c})^c. \end{aligned}$$

(b) Using (a), we see that $\partial A = \overline{A} \cap \overline{A^c} = \overline{A} \setminus (\overline{A^c})^c = \overline{A} \setminus A^o$.
(c) If $x \in (A \setminus A^o)^o$, then for some open set V we have

$$x \in V \subseteq A \setminus A^o \subseteq A.$$

This implies $x \in A \setminus A^o$ and $x \in A^o$, a contradiction. Hence, $(A \setminus A^o)^o = \emptyset$.

Problem 8.2. *If A and B are two arbitrary subsets of a topological space, then show the following:*

a. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
b. $(A \cup B)' = A' \cup B'$.

Solution. (a) See Problem 6.1.
(b) Clearly, $A \subseteq B$ implies $A' \subseteq B'$, and so $A' \cup B' \subseteq (A \cup B)'$. For the reverse inclusion, let $x \in (A \cup B)'$. If $x \notin A' \cup B'$, then there exist two neighborhoods

V and W of x such that

$$A \cap (V \setminus \{x\}) = B \cap (W \setminus \{x\}) = \emptyset.$$

Now, note that the neighborhood $U = V \cap W$ of the point x satisfies

$$(A \cup B) \cap (U \setminus \{x\}) = \emptyset,$$

proving that $x \notin (A \cup B)'$, a contradiction.

Problem 8.3. *If A is an arbitrary subset of a Hausdorff topological space, then show that its derived set A' is a closed set.*

Solution. Let A be an arbitrary subset of a Hausdorff topological space X. We shall establish that $(A')^c$ is an open set (and this will guarantee that A' is a closed set). To this end, let $x \in (A')^c$, i.e., let $x \notin A'$. This means that there exists a neighborhood V of x such that

$$V \cap \big(A \setminus \{x\}\big) = \emptyset. \tag{$\star$}$$

We claim that $V \subseteq (A')^c$ holds. To see this, let $y \in V$ with $y \neq x$. Since X is a Hausdorff topological space, there exist neighborhoods U and W of y and x, respectively, such that $U \cap W = \emptyset$. Now, note that $V \cap U$ is a neighborhood of y with $x \notin V \cap U$ and so from ($\star$), we see that $(V \cap U) \cap A = \emptyset$. The latter shows that $y \notin A'$. Hence, $V \subseteq (A')^c$ holds proving that every point of $(A')^c$ is an interior point, as desired.

Problem 8.4. *Let $X = \mathbb{R}$, and let τ be the topology on X defined in Example 8.4. In other words, $A \in \tau$ if and only if for each $x \in A$ there exist $\epsilon > 0$ and an at-most countable set B (both depending on x) such that $(x - \epsilon, x + \epsilon) \setminus B \subseteq A$.*

a. *Show that τ is a topology on X.*
b. *Verify that $0 \in \overline{(0, 1)}$.*
c. *Show that there is no sequence $\{x_n\}$ of $(0, 1)$ with $\lim x_n = 0$.*

Solution. (a) Straightforward.

(b) Since for each $\varepsilon > 0$ and each countable set B the set $(-\varepsilon, \varepsilon) \setminus B$ is uncountable, we must have $\big((-\varepsilon, \varepsilon) \setminus B\big) \cap (0, 1) \neq \emptyset$. This easily implies that $0 \in \overline{(0, 1)}$.

(c) If $\{x_n\}$ is a sequence of $(0, 1)$, then $V = (-1, 1) \setminus \{x_1, x_2, \ldots\}$ is a neighborhood of zero, and $x_n \notin V$ for all n. This shows that no sequence of $(0, 1)$ can converge to 0.

Problem 8.5. *If A is a dense subset of a topological space, then show that $\mathcal{O} \subseteq \overline{A \cap \mathcal{O}}$ holds for every open set $\mathcal{O}$. Generalize this conclusion as follows: If A is open, then $A \cap \overline{B} \subseteq \overline{A \cap B}$ for each set B.*

Solution. Let $x \in \mathcal{O}$ and let V be a neighborhood of x. Since $\mathcal{O}$ is open, $V \cap \mathcal{O}$ is a neighborhood of x, and so the denseness of A implies

$$V \cap (A \cap \mathcal{O}) = (V \cap \mathcal{O}) \cap A \neq \emptyset,$$

which means that $x \in \overline{A \cap \mathcal{O}}$.

For the general case, assume A is an open set and let $x \in A \cap \overline{B}$. If V is a neighborhood of x, then $V \cap A$ is also a neighborhood of x. Since $x \in \overline{B}$, it follows that $V \cap (A \cap B) = (V \cap A) \cap B \neq \emptyset$. This shows that $x \in \overline{A \cap B}$, and hence, $A \cap \overline{B} \subseteq \overline{A \cap B}$.

Problem 8.6. *If $\{\mathcal{O}_i\}_{i \in I}$ is an open cover for a topological space X, then show that a subset A of X is closed if and only if $A \cap \mathcal{O}_i$ is closed in $\mathcal{O}_i$ for each $i \in I$ (where $\mathcal{O}_i$ is considered equipped with the relative topology).*

Solution. If A is closed, then clearly $A \cap \mathcal{O}_i$ is closed in $\mathcal{O}_i$ for each i. For the converse, assume that $A \cap \mathcal{O}_i$ is closed in $\mathcal{O}_i$ for each i. Put

$$V_i = \mathcal{O}_i \setminus A \cap \mathcal{O}_i = \mathcal{O}_i \setminus A,$$

and note that—by our hypothesis—each V_i is open in $\mathcal{O}_i$. Since each $\mathcal{O}_i$ is an open subset of X, it follows that each V_i is likewise an open subset of X. Now, note that

$$A^c = X \setminus A = \Big(\bigcup_{i \in I} \mathcal{O}_i\Big) \setminus A = \bigcup_{i \in I} (\mathcal{O}_i \setminus A) = \bigcup_{i \in I} V_i$$

is an open subset of X, and so A is a closed set.

Problem 8.7. *If (X, τ) is a Hausdorff topological space, then show the following:*

a. *Every finite subset of X is closed.*
b. *Every sequence of X converges to at-most one point.*

Solution. (a) Let $A = \{x\}$ be a one-point set. If $y \notin A$, then (since X is a Hausdorff space) there exists a neighborhood V of y with $x \notin V$, and so $V \subseteq A^c$. Thus, A^c is open, and hence A, is closed. Now, observe that every finite set is a finite union of one-point sets.

(b) If $x \neq y$, then there exist neighborhoods V_x and V_y of x and y respectively, such that $V_x \cap V_y = \emptyset$. Now, a sequence of X cannot converge to x and y at the same time simply because its terms cannot be eventually in both V_x and V_y.

Problem 8.8. *For a function $f:(X,\tau) \to (Y,\tau_1)$ show the following:*

a. *If τ is the discrete topology, then f is continuous.*
b. *If τ is the indiscrete topology and τ_1 is a Hausdorff topology, then f is continuous if and only if f is a constant function.*

Solution. (a) Note that every subset of X is open. Thus, $f^{-1}(A)$ is an open set for every subset A of Y, and so f is continuous.

(b) Recall that the indiscrete topology is the topology $\tau = \{\emptyset, X\}$. If f is a constant function, then $f^{-1}(A)$ is either $\emptyset$ or X, and so f is continuous. For the converse, let f be a continuous function. If for some $x, y \in X$ we have $f(x) \neq f(y)$, then there exists a neighborhood V of $f(x)$ such that $f(y) \notin V$. Now note that $f^{-1}(V)$ is neither equal to $\emptyset$ nor equal to X, and so $f^{-1}(V)$ is not open, a contradiction. Thus, f must be a constant function.

Problem 8.9. *Let f and g be two continuous functions from (X,τ) into a Hausdorff topological space (Y,τ_1). Assume that there exists a dense subset A of X such that $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ holds for all $x \in X$.*

Solution. Suppose that for some $x \in X$ we have $f(x) \neq g(x)$. Pick a neighborhood V of $f(x)$ and another W of $g(x)$ such that $V \cap W = \emptyset$. Since $f^{-1}(V) \cap g^{-1}(W)$ is a neighborhood of x and A is dense in X, there exists some $y \in f^{-1}(V) \cap g^{-1}(W) \cap A$. Now, note that $f(y) = g(y) \in V \cap W = \emptyset$ must hold, which is absurd. Thus, $f(x) = g(x)$ holds for each $x \in X$.

Problem 8.10. *Let $f:(X,\tau) \to (Y,\tau_1)$ be a function. Show that f is continuous if and only if $f^{-1}(B^o) \subseteq [f^{-1}(B)]^o$ holds for every subset B of Y.*

Solution. Repeat the solution of Problem 6.4.

Problem 8.11. *If $f:(X,\tau) \to (Y,\tau_1)$ and $g:(Y,\tau_1) \to (Z,\tau_2)$ are continuous functions, show that their composition $g \circ f:(X,\tau) \to (Z,\tau_2)$ is also continuous.*

Solution. Use the identity $(g \circ f)^{-1}(V) = f^{-1}\big(g^{-1}(V)\big)$. (See Problem 1.8.)

Problem 8.12. *Let X be a topological space, let $a \in X$, and let $\mathcal{N}_a$ denote the collection of all neighborhoods at a. The* **oscillation** *of a function $f: X \to \mathbf{R}$ at the point a is the extended non-negative real number*

$$\omega_f(a) = \inf_{V \in \mathcal{N}_a} \Big\{ \sup_{x,y \in V} |f(x) - f(y)| \Big\}.$$

Establish the following properties regarding the oscillation:

a. *The function f is continuous at a if and only if $\omega_f(a) = 0$.*
b. *If X is an open interval of $\mathbb{R}$ and $f: X \to \mathbb{R}$ is a monotone function, then $\omega_f(a) = \left| \lim_{x \to a^+} f(x) - \lim_{x \to a^-} f(x) \right|$.*

Solution. (a) Assume that f is continuous at a. Fix $\epsilon > 0$. Then there exists some $W \in \mathcal{N}_a$ (i.e., some neighborhood W of a) such that $x \in W$ implies $|f(a) - f(x)| < \epsilon$. So, if $x, y \in W$, then

$$|f(x) - f(y)| \le |f(x) - f(a)| + |f(a) - f(y)| < \epsilon + \epsilon = 2\epsilon,$$

and thus

$$0 \le \omega_f(a) \le \sup_{x,y \in W} \left| f(x) - f(y) \right| \le 2\epsilon$$

for each $\epsilon > 0$. This implies $\omega_f(a) = 0$.

For the converse, assume $\omega_f(a) = 0$. Let $\epsilon > 0$. Then from the definition of the oscillation, we see that there exists some neighborhood V of a such that $\sup_{x,y \in V} \left| f(x) - f(y) \right| < \epsilon$. In particular, we have $|f(x) - f(a)| < \epsilon$ for all $x \in V$, and this shows that f is continuous at a.

(b) We can assume that f is an increasing function. Note that we can consider neighborhoods of a of the form (c, d) with $a \in (c, d)$. Consider first a neighborhood (c, d) of a and assume that $x, y \in (c, d)$ satisfy $x < a < y$. Since f is increasing, it follows that $0 \le \lim_{t \to a^+} f(t) - \lim_{t \to a^-} f(t) \le f(y) - f(x)$, and from this, we infer that

$$\lim_{t \to a^+} f(t) - \lim_{t \to a^-} f(t) \le \omega_f(a).$$

On the other hand, if $\epsilon > 0$ is given, then there exists some $\delta > 0$ such that the open interval $J = (a - \delta, a + \delta)$ satisfies $J \subseteq X$ and

$$\omega_f(a) \le \sup_{x,y \in J} |f(x) - f(y)| < \left[\lim_{t \to a^+} f(t) - \lim_{t \to a^-} f(t) \right] + \epsilon.$$

This implies $\omega_f(a) \le \lim_{t \to a^+} f(t) - \lim_{t \to a^-} f(t)$, and so

$$\omega_f(a) = \lim_{t \to a^+} f(t) - \lim_{t \to a^-} f(t)$$

holds true.

Problem 8.13. *Show that a finite union of nowhere dense sets is again a nowhere dense set. Is this statement true for a countable union of nowhere dense sets?*

Solution. Let A and B be two nowhere dense sets. Using the identity $S^{o} = S^{c-c}$ (see Problem 8.1), we have

$$\begin{aligned}(\overline{A \cup B})^{o} &= (\overline{A} \cup \overline{B})^{c-c} = (A^{-c} \cap B^{-c})^{-c} \subseteq (A^{-c-} \cap B^{-c-})^{c} \\ &= A^{-c-c} \cup B^{-c-c} = (\overline{A})^{o} \cup (\overline{B})^{o} = \emptyset \cup \emptyset = \emptyset.\end{aligned}$$

An easy induction argument can now complete the proof.

The countable union of nowhere dense sets need not be nowhere dense. An example: Take $X = \mathbf{R}$, and let $E_n = \{r_n\}$, where $\{r_1, r_2, \ldots\}$ is an enumeration of the rational numbers. Clearly, each E_n is nowhere dense, while $\bigcup_{n=1}^{\infty} E_n = \{r_1, r_2, \ldots\}$ is not nowhere dense.

Problem 8.14. *Show that the boundary of an open or closed set is nowhere dense.*

Solution. Repeat the solution of Problem 6.5.

Problem 8.15. *Let $f: (X, \tau) \to \mathbf{R}$, and let D be the set of all points of X where f is discontinuous. If D^{c} is dense in X, then show that D is a meager set.*

Solution. From $\overline{D^{c}} = X$, it follows that $D^{o} = \left(\overline{D^{c}}\right)^{c} = \emptyset$. Now, the proof can be completed by observing that D is an F_{σ}-set (Theorem 8.10).

Problem 8.16. *Show that there is no function $f: \mathbf{R} \to \mathbf{R}$ having the irrational numbers as the set of its discontinuities.*

Solution. Let I denote the set of all irrational numbers of $\mathbf{R}$. If I is the set of discontinuities of a function $f: \mathbf{R} \longrightarrow \mathbf{R}$, then (by Theorem 8.10) I is an F_{σ}-set. However, this is impossible by Problem 6.6.

Problem 8.17. *Show that every closed subset of a metric space is a G_{δ}-set and every open set is an F_{σ}-set.*

Solution. Let A be a nonempty closed subset of a metric space X. Then the function $f: X \longrightarrow \mathbf{R}$, defined by

$$f(x) = d(x, A) = \inf\{d(x, y)\colon\ y \in A\},$$

is continuous (see the proof of Lemma 10.4) and satisfies

$$A = f^{-1}(\{0\}) = f^{-1}\Big(\bigcap_{n=1}^{\infty}\left(-\tfrac{1}{n}, \tfrac{1}{n}\right)\Big) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left(-\tfrac{1}{n}, \tfrac{1}{n}\right)\right).$$

(See the discussion at the end of Section 6 of the text.) Thus, A is a G_δ-set. By Theorem 8.9 every open set is an F_σ-set.

Problem 8.18. *Let $\mathcal{B}$ be a collection of open sets in a topological space (X, τ). If for each x in an arbitrary open set V there exists some $B \in \mathcal{B}$ with $x \in B \subseteq V$, then $\mathcal{B}$ is called a* **base** *for τ. In general, a collection $\mathcal{B}$ of subsets of a nonempty set X is said to be a* **base** *if*

1. $\bigcup_{B \in \mathcal{B}} B = X$, *and*
2. *for every pair $A, B \in \mathcal{B}$ and $x \in A \cap B$, there exists some $C \in \mathcal{B}$ with $x \in C \subseteq A \cap B$.*

Show that if $\mathcal{B}$ is a base for a set X, then the collection

$$\tau = \{V \subseteq X\colon \forall\, x \in V \text{ there exists } B \in \mathcal{B} \text{ with } x \in B \subseteq V\}$$

is a topology on X having $\mathcal{B}$ as a base.

Solution. Obviously, $\mathcal{B} \subseteq \tau$ holds. Clearly, $\emptyset \in \tau$, and from condition (1) it follows that $X \in \tau$. Also, it should be clear that τ is closed under arbitrary unions.

Now, let $V, W \in \tau$ and $x \in V \cap W$. Choose two sets $A, B \in \mathcal{B}$ with $x \in A \subseteq V$ and $x \in B \subseteq W$. By condition (2), there exists some $C \in \mathcal{B}$ with $x \in C \subseteq A \cap B \subseteq V \cap W$, that is, $V \cap W \in \tau$. Thus, τ is a topology.

The verification that $\mathcal{B}$ is a base for τ is straightforward.

Problem 8.19. *Let (X, τ) be a topological space, and let $\mathcal{B}$ be a base for the topology τ (see the preceding exercise for the definition). Show that there exists a dense subset A of X such that* $\operatorname{card} A \leq \operatorname{card} \mathcal{B}$.

Solution. If $B \in \mathcal{B}$ and $B \neq \emptyset$, then fix some $x_B \in B$ and consider the set $A = \{x_B\colon B \in \mathcal{B} \setminus \{\emptyset\}\}$. We claim that:

1. A is dense in X, and
2. $\operatorname{card} A \leq \operatorname{card} \mathcal{B}$.

To see (1) let V be a nonempty open set. If $x \in V$, then there exists some $B \in \mathcal{B}$ with $x \in B \subseteq V$. It follows that $x_B \in V$, and so $V \cap A \neq \emptyset$. This shows that A is dense in X.

For (2) note that the function $f\colon \mathcal{B} \setminus \{\emptyset\} \longrightarrow A$, defined by $f(B) = x_B$, is onto. By the Axiom of Choice there exists a subset $\mathcal{C}$ of $\mathcal{B}$ such that $\mathcal{C} \cap f^{-1}(\{x\})$ consists precisely of one point for each $x \in A$. Then $f\colon \mathcal{C} \longrightarrow A$ is one-to-one and onto, proving that $\operatorname{card} A = \operatorname{card} \mathcal{C} \leq \operatorname{card} \mathcal{B}$.

Problem 8.20. *Let $f: X \to Y$ be a function. If τ is a topology on X, then the* **quotient topology** τ_f *determined by f on Y is defined by* $\tau_f = \{O \subseteq Y: f^{-1}(O) \in \tau\}$.

a. *Show that τ_f is indeed a topology on Y and that $f: (X, \tau) \to (Y, \tau_f)$ is continuous.*
b. *If $g: (Y, \tau_f) \to (Z, \tau_1)$ is a function, then show that the composition function $g \circ f: (X, \tau) \to (Z, \tau_1)$ is continuous if and only if g is continuous.*
c. *Assume that $f: X \to Y$ is onto and that τ^* is a topology on Y such that $f: (X, \tau) \to (Y, \tau^*)$ is an open mapping (i.e., it carries open sets of X onto open sets of Y) and continuous. Show that $\tau^* = \tau_f$.*

Solution. a. (1) Since $f^{-1}(\emptyset) = \emptyset \in \tau$ and $f^{-1}(Y) = X \in \tau$, we see that $\emptyset, Y \in \tau_f$.
(2) If $V, W \in \tau_f$, then the identity $f^{-1}(V \cap W) = f^{-1}(V) \cap f^{-1}(W)$ implies that $V \cap W \in \tau_f$.
(3) If $\{V_i: i \in I\}$ is a family of τ_f, then in view of the identity $f^{-1}\left(\bigcup V_i\right) = \bigcup f^{-1}(V_i)$, we see that $\bigcup V_i \in \tau_f$.
b. Assume $g \circ f$ is continuous. If V is an open subset of Z, then $f^{-1}\left(g^{-1}(V)\right) = (g \circ f)^{-1}(V) \in \tau$ shows that $g^{-1}(V) \in \tau_f$. That is, g is continuous.
c. Since f is continuous, it is easy to see that $\tau^* \subseteq \tau_f$ holds. On the other hand, let $V \in \tau_f$. Then $f^{-1}(V) \in \tau$, and moreover, since f is an open mapping and onto, we have $V = f\left(f^{-1}(V)\right) \in \tau^*$ (see Problem 1.7). That is, $\tau_f \subseteq \tau^*$ also holds, and so $\tau_f = \tau^*$.

Problem 8.21. *This exercise presents an example of a compact set whose closure is not compact. Start by considering the interval $[0, 1]$ with the topology τ generated by the metric $d(x, y) = |x - y|$. It should be clear that $([0, 1], \tau)$ is a compact topological space. Next, put $X = [0, 1] \cup \mathbb{N} = [0, 1] \cup \{2, 3, 4, \ldots\}$, and define*

$$\tau^* = \tau \cup \{[0, 1] \cup A: A \subseteq \mathbb{N}\}.$$

a. *Show that τ^* is a non-Hausdorff topology on X and that τ^* induces τ on $[0, 1]$.*
b. *Show that (X, τ^*) is not a compact topological space.*
c. *Show that $[0, 1]$ is a compact subset of (X, τ^*).*
d. *Show that $[0, 1]$ is dense in X (and hence, its closure is not compact.*
e. *Why doesn't this contradict Theorem 8.12(1)?*

Solution. a. (1) Clearly, $\emptyset, X \in \tau^*$.
(2) Let $V, W \in \tau^*$. Then we have the following cases:
CASE I. $V, W \in \tau$. In this case, $V \cap W \in \tau \subseteq \tau^*$.
CASE II. $V \in \tau$ and $W \notin \tau$ (and vice versa). Note that $V \cap W = V \in \tau \subseteq \tau^*$.

CASE III. $V \notin \tau$ and $W \notin \tau$. In this case, we have $V \cap W = [0, 1] \cup A$ for some $A \subseteq \mathcal{N}$. That is, $V \cap W \in \tau^*$.
(3) Let $\{V_i: i \in I\}$ be a family of τ^*. If $V_i \in \tau$ holds for each i, then clearly $\bigcup V_i \in \tau \subseteq \tau^*$ holds. On the other hand, if some V_i is of the form $[0, 1] \cup A$, then $\bigcup V_i$ is of the same type, and hence, it belongs to τ^*.

Thus, τ^* is a topology on X that induces τ on $[0, 1]$.
b. The cover $X = \bigcup_{n=2}^{\infty}\big([0, 1] \cup \{n\}\big)$ cannot be reduced to a finite cover.
c. Since $([0, 1], \tau)$ is a compact topological space and τ^* induces τ on $[0, 1]$, it follows that $[0, 1]$ is a compact subset of X.
d. If $x \in X \setminus [0, 1]$, then every neighborhood V of x is of the form $V = [0, 1] \cup A$ for some $A \subseteq \mathcal{N}$. Thus, $V \cap [0, 1] = [0, 1] \neq \emptyset$ holds for every neighborhood V of x. Therefore, $\overline{[0, 1]} = X$ holds.
e. This does not contradict Theorem 8.12(1) because (X, τ^*) is not a Hausdorff topological space.

Problem 8.22. *A topological space (X, τ) is said to be* **connected** *if a subset of X that is simultaneously closed and open (called a* **clopen set**) *is either empty or else equal to X.*

a. *Show that (X, τ) is connected if and only if the only continuous functions from (X, τ) into $\{0, 1\}$ (with the discrete topology) are the constant ones.*
b. *Let $f:(X, \tau) \to (Y, \tau^*)$ be onto and continuous. If (X, τ) is connected, then show that (Y, τ^*) is also connected.*

Solution. (a) If $f: X \longrightarrow \{0, 1\}$ is a nonconstant continuous function, then $f^{-1}(\{0\})$ is a nonempty clopen set which is different from X, and so X is not connected.

For the converse, assume that every continuous function from X into $\{0, 1\}$ is constant. If A is a clopen subset of X different from $\emptyset$ and X, then the function $f: X \longrightarrow \{0, 1\}$, defined by $f(x) = 1$ if $x \in A$ and $f(x) = 0$ if $x \notin A$, is a nonconstant continuous function, a contradiction. Thus, X is a connected topological space.

(b) Let A be a clopen subset of Y. By the continuity of f, the set $f^{-1}(A)$ is a clopen subset of X. Since X is connected, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = X$. Also, since f is onto, $f\big(f^{-1}(A)\big) = A$ holds (Problem 1.7). Thus, $A = \emptyset$ or $A = Y$, proving that Y is a connected topological space.

9. CONTINUOUS REAL-VALUED FUNCTIONS

Problem 9.1. *If u, v, and w are vectors in a vector lattice, then establish the following identities:*

a. $u \vee v + u \wedge v = u + v$;

b. $u - v \vee w = (u - v) \wedge (u - w)$;
c. $u - v \wedge w = (u - v) \vee (u - w)$;
d. $\alpha(u \wedge v) = (\alpha u) \wedge (\alpha v)$ *if* $\alpha \geq 0$;
e. $|u - v| = u \vee v - u \wedge v$;
f. $u \vee v = \frac{1}{2}(u + v + |u - v|)$;
g. $u \wedge v = \frac{1}{2}(u + v - |u - v|)$.

Solution. We use the identities (a), (b), and (d) in Section 9 of the text.
(a) Replace w by $-(u + v)$ in $u \wedge v + w = (u + w) \wedge (v + w)$ to get

$$u \wedge v - (u + v) = (-v) \wedge (-u) = -u \vee v.$$

(b) $u - v \vee w = u + (-v) \wedge (-w) = (u - v) \wedge (u - w)$.
(c) $u - v \wedge w = u + (-v) \vee (-w) = (u - v) \vee (u - w)$.
(d) If $\alpha \geq 0$, then

$$\begin{aligned} \alpha(u \wedge v) &= \alpha[-(-u) \vee (-v)] = -\alpha[(-u) \vee (-v)] \\ &= -(-\alpha u) \vee (-\alpha v) = (\alpha u) \wedge (\alpha v). \end{aligned}$$

(e) Using (a), we see that

$$\begin{aligned} u \vee v - u \wedge v &= u \vee v + \big[u \vee v - (u + v)\big] = 2(u \vee v) - (u + v) \\ &= (2u) \vee (2v) - (u + v) = (u - v) \vee (v - u) = |u - v|. \end{aligned}$$

(f) Using (e) and (a), we get

$$u + v + |u - v| = \big(u \vee v + u \wedge v\big) + \big(u \vee v - u \wedge v\big) = 2(u \vee v).$$

(g) As in (f), we get

$$u + v - |u - v| = u \vee v + u \wedge v - (u \vee v - u \wedge v) = 2(u \wedge v).$$

Problem 9.2. *If u and v are elements in a vector lattice, then show that:*
a. $|u + v| \vee |u - v| = |u| + |v|$, *and*
b. $|u + v| \wedge |u - v| = \big||u| - |v|\big|$.

Solution. (a) Note that

$$\begin{aligned} |u + v| \vee |u - v| &= [(u + v) \vee (-u - v)] \vee [(u - v) \vee (-u + v)] \\ &= [(u + v) \vee (-u + v)] \vee [(-u - v) \vee (u - v)] \\ &= [u \vee (-u) + v] \vee [(-u) \vee u - v] \\ &= [u \vee (-u)] + [v \vee (-v)] = |u| + |v|. \end{aligned}$$

(b) Using the distributive law, we see that

$$
\begin{aligned}
&|u+v| \wedge |u-v| = [(u+v) \vee (-u-v)] \wedge [(u-v) \vee (-u+v)] \\
&= [(u+v) \wedge (u-v)] \vee [(-u-v) \wedge (u-v)] \vee [(u+v) \wedge (-u+v)] \vee \cdots \\
&\qquad \cdots \vee [(-u-v) \wedge (-u+v)] \\
&= [u + v \wedge (-v)] \vee [(-u) \wedge u - v] \vee [u \wedge (-u) + v] \vee [v \wedge (-v) - u] \\
&= (u - |v|) \vee (-u - |v|) \vee (v - |u|) \vee (-v - |u|) \\
&= [u \vee (-u) - |v|] \vee [v \vee (-v) - |u|] = (|u| - |v|) \vee (|v| - |u|) \\
&= \big| |u| - |v| \big|.
\end{aligned}
$$

Problem 9.3. *Show that* $|u| \wedge |v| = 0$ *holds if and only if* $|u+v| = |u-v|$ *holds.*

Solution. If $|u| \wedge |v| = 0$, then using parts (a) and (b) and part (e) of Problem 9.1, we get

$$
\begin{aligned}
|u+v| \wedge |u-v| &= \big||u| - |v|\big| = |u| \vee |v| - |u| \wedge |v| = |u| \vee |v| \\
&= |u| + |v| - |u| \wedge |v| = |u| + |v| = |u+v| \vee |u-v|.
\end{aligned}
$$

This easily implies that $|u+v| = |u-v|$ holds.

For the converse, assume that $|u+v| = |u-v|$. Then by parts (a) and (b) of Problem 9.2, we have

$$
\begin{aligned}
|u| + |v| &= \big||u| - |v|\big| = |u| \vee |v| - |u| \wedge |v| \\
&= (|u| + |v| - |u| \wedge |v|) - |u| \wedge |v| = |u| + |v| - 2(|u| \wedge |v|),
\end{aligned}
$$

from which it follows that $|u| \wedge |v| = 0$.

Problem 9.4. *Show that the vector space consisting of all polynomials (with real coefficients) on* $\mathbf{R}$ *is not a function space. Prove a similar result for the vector space of all real-valued differentiable functions on* $\mathbf{R}$.

Solution. If p is the polynomial defined by $p(x) = x$, then $|p|(x) = |p(x)| = |x|$ holds. Clearly, $|p|$ is not differentiable (and hence, it is not a polynomial either).

Problem 9.5. *Let X be a topological space. Consider the collection L of all real-valued functions on X defined by*

$$
L = \{f \in \mathbf{R}^X \colon \exists \, \{f_n\} \subseteq C(X) \text{ such that } \lim f_n(x) = f(x) \; \forall \, x \in X\}.
$$

Show that L is a function space.

Solution. Clearly, L is a vector space. Now, let $f, g \in L$. Choose two sequences $\{f_n\}$ and $\{g_n\}$ of $C(X)$ with $\lim f_n(x) = f(x)$ and $\lim g_n(x) = g(x)$ for all x. Then $f_n \vee g_n \in C(X)$ for each n and

$$\begin{aligned}\lim f_n \vee g_n(x) &= \lim \frac{1}{2}\big[f_n(x) + g_n(x) + |f_n(x) - g_n(x)|\big] \\ &= \frac{1}{2}\big[f(x) + g(x) + |f(x) - g(x)|\big] = f \vee g(x),\end{aligned}$$

so that $f \vee g \in L$. Similarly, $f \wedge g \in L$, so that L is a function space.

Problem 9.6. *Let L be a vector space of real-valued functions defined on a set X. If for every function $f \in L$ the function $|f|$ (defined by $|f|(x) = |f(x)|$ for each $x \in X$) belongs to L, then show that L is a function space.*

Solution. Use the identities

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \quad \text{and} \quad f \wedge g = \frac{1}{2}(f + g - |f - g|).$$

Problem 9.7. *Consider each rational number written in the form $\frac{m}{n}$, where $n > 0$, and m and n are integers without any common factors other than ± 1. Clearly, such a representation is unique. Now, define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ as above. Show that f is continuous at every irrational number and discontinuous at every rational number.*

Solution. The proof will be based upon the following property: *Let $\{r_n\}$ be a bounded sequence of distinct rational numbers. If $r_n = \frac{m_n}{k_n}$ (where $k_n > 0$, and m_n and k_n do not have common factors), then* $\lim k_n = \infty$.

To see this, pick some number $M > 0$ such that $|r_n| \le M$ for each n, and so $|m_n| \le Mk_n$. Now, if for some $C > 0$ we have $|k_n| \le C$ for infinitely many n, then $|m_n| \le MC$ must also hold for the same infinitely many n. However, this contradicts the fact that there is a finite number of rational numbers $\frac{m}{n}$ with $|m| \le MC$ and $|n| < C$.

Now, let x be an irrational number. If $\{x_n\}$ is a sequence of irrational numbers with $x_n \longrightarrow x$, then $0 = f(x_n) \longrightarrow 0 = f(x)$. Thus, if f is not continuous at x, then there exists a sequence $\{r_n\}$ of rational numbers with $r_n \longrightarrow x$ and $\lim f(r_n) \neq 0$. Since x is irrational, we can assume $r_n \neq r_m$ whenever $n \neq m$. Write $r_n = \frac{m_n}{k_n}$, and note that $f(r_n) = \frac{1}{k_n} \not\to 0$ implies $k_n \not\to \infty$, a contradiction. Therefore, f is continuous at every irrational number.

Now, let r be a rational number. Choose a sequence $\{r_n\}$ of distinct rational numbers with $r_n = \frac{m_n}{k_n} \longrightarrow r$. Now, note that $\lim f(r_n) = \lim \frac{1}{k_n} = 0 \neq f(r)$ holds, which shows that f is not continuous at r. That is, f is discontinuous at every rational number.

Problem 9.8. *Let $f: [a, b] \to \mathbf{R}$ be increasing, i.e., $x < y$ implies $f(x) \leq f(y)$. Show that the set of points where f is discontinuous is at-most countable.*

Solution. Let $f: [a, b] \longrightarrow \mathbf{R}$ be increasing, and let D be the set of discontinuities of f. For each $x \in D$ choose a rational number r_x such that $\lim_{t \uparrow x} f(t) < r_x < \lim_{t \downarrow x} f(t)$. Since $x, y \in D$ with $x < y$ implies

$$r_x < \lim_{t \downarrow x} f(t) < \lim_{t \uparrow y} f(t) < r_y,$$

it follows that $r_x \neq r_y$ whenever $x \neq y$. Thus, $x \longmapsto r_x$ is a one-to-one function from D into the set of rational numbers, and so D is at-most countable.

Problem 9.9. *Give an example of a strictly increasing function $f: [0, 1] \to \mathbf{R}$ which is continuous at every irrational number and discontinuous at every rational number.*

Solution. For each $t \in [0, 1]$, let $f_t: [0, 1] \to [0, 1]$ be a strictly increasing function which is continuous everywhere except at $x = t$. For instance, for $0 < t \leq 1$ let

$$f_t(x) = \begin{cases} 0.5x & \text{if } 0 \leq x < t, \\ x & \text{if } t \leq x \leq 1, \end{cases}$$

and $f_0(x) = 0.5 + 0.5x$ if $0 < x \leq 1$ and $f_0(0) = 0$.

If $\{r_1, r_2, \ldots\}$ is an enumeration of the rational numbers of $[0, 1]$, then define the function $f: [0, 1] \to [0, 1]$ by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{r_n}(x),$$

and note that f satisfies the desired properties.

Problem 9.10. *Recall that a function $f: (X, \tau) \to (Y, \tau_1)$ is called an open mapping if $f(V)$ is open whenever V is open. Prove that if $f: \mathbf{R} \to \mathbf{R}$ is a continuous open mapping, then f is a strictly monotone function—and hence, a homeomorphism.*

Solution. Let (a, b) be a finite open interval of $\mathbf{R}$. Since f attains a maximum value on $[a, b]$ and $f\big((a, b)\big)$ is an open set, it is easy to see that the extrema of f on $[a, b]$ take place at the end points. In particular, this implies $f(a) \neq f(b)$. (If $f(a) = f(b)$, then $f\big((a, b)\big)$ must be a one-point set, contradicting the fact that f is an open mapping.) Next, we claim that f is strictly monotone on (a, b). To see this, assume $f(a) < f(b)$, and $a < x < y < b$. Then note first that

$f(a) < f(x) < f(b)$ must hold. Indeed, if $f(x) \leq f(a)$ holds, then f attains its minimum on $[a, b]$ at some interior point. Similarly, if $f(x) \geq f(b)$ holds, then f attains its maximum value on $[a, b]$ at some interior point. However, (since f is an open mapping) both cases are impossible, and so $f(a) < f(x) < f(b)$ holds. By the same arguments, $f(x) < f(y) < f(b)$. Thus, f is strictly increasing on (a, b). Similarly, if $f(a) > f(b)$ holds, then f is strictly decreasing on (a, b).

Now, assume that f is strictly increasing on $(0, 1)$, and let $x < y$. Choose some n with $(0, 1) \subseteq (-n, n)$ and $x, y \in (-n, n)$. Since f is strictly monotone on $(-n, n)$, and strictly increasing on $(0, 1)$, it is easy to see that f must be strictly increasing on $(-n, n)$. Thus, $f(x) < f(y)$ holds, and this shows that f is strictly increasing on $\mathbf{R}$. (We remark that the function f need not be onto. However, the mapping $f: \mathbf{R} \longrightarrow f(\mathbf{R})$ is a homeomorphism.)

Problem 9.11. *Let X be a nonempty set, and for any two functions $f, g \in \mathbf{R}^X$ let*

$$d(f, g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}.$$

Establish the following:

a. *$(\mathbf{R}^X, d)$ is a metric space.*
b. *A sequence $\{f_n\} \subseteq \mathbf{R}^X$ satisfies $d(f_n, f) \to 0$ for some $f \in \mathbf{R}^X$ if and only if $\{f_n\}$ converges uniformly to f.*

Solution. (a) Clearly, $d(f, g) \geq 0$ for all $f, g \in \mathbf{R}^X$ and $d(f, g) = 0$ if and only if $f = g$. Moreover, it should be clear that $d(f, g) = d(g, f)$ for all $f, g \in \mathbf{R}^X$. What needs verification is the triangle inequality. To do this, we need the following two properties:

1. $0 \leq x \leq y$ implies $\frac{x}{1+x} \leq \frac{y}{1+y}$, and
2. $\frac{x+y}{1+x+y} \leq \frac{x}{1+x} + \frac{y}{1+y}$ for all $x, y \geq 0$.

Property (1) follows from the fact that the function $f(t) = \frac{t}{1+t}$ $(t \geq 0)$ is strictly increasing on $[0, \infty)$; notice that $f'(t) = (1+t)^{-2} > 0$ for each $t > -1$. For (2) fix $x, y \geq 0$, and note that

$$\begin{aligned}(x+y)(1+x)(1+y) &= x(1+x)(1+y) + y(1+x)(1+y)\\ &\leq \big[x(1+x)(1+y) + xy(1+y)\big] + \big[y(1+x)(1+y) + xy(1+x)\big]\\ &= x(1+y)(1+x+y) + y(1+x)(1+x+y).\end{aligned}$$

Dividing across by $(1+x)(1+y)(1+x+y)$, the validity of (2) can be established.

Now, let $f, g, h \in \mathbf{R}^X$ and $x \in X$. From

$$|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$$

and (1) and (2), we get

$$\begin{aligned}\frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} &\leq \frac{|f(x) - h(x)| + |h(x) - g(x)|}{1 + |f(x) - h(x)| + |h(x) - g(x)|} \\ &\leq \frac{|f(x) - h(x)|}{1 + |f(x) - h(x)|} + \frac{|h(x) - g(x)|}{1 + |h(x) - g(x)|} \\ &\leq d(f, h) + d(h, g),\end{aligned}$$

for all $x \in X$. This implies

$$d(f, g) = \sup_{x \in X} \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \leq d(f, h) + d(h, g).$$

(b) Let $\{f_n\} \subseteq \mathbf{R}^X$. Assume first that $\{f_n\}$ converges uniformly to some function $f \in \mathbf{R}^X$, and let $\epsilon > 0$. So, there exists n_0 such that $|f_n(x) - f(x)| < \epsilon$ holds for all $n \geq n_0$ and all $x \in X$, and hence, $\frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \leq |f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$ and all $x \in X$. It follows that

$$d(f_n, f) = \sup_{x \in X} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \leq \epsilon$$

for all $n \geq n_0$. This shows that $d(f_n, f) \to 0$.

For the converse, assume $d(f_n, f) \to 0$, and let $\epsilon > 0$. Then there exists some n_0 such that

$$d(f_n, f) = \sup_{x \in X} \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} < \frac{\epsilon}{1 + \epsilon}$$

for all $n \geq n_0$, and hence, $\frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} < \frac{\epsilon}{1+\epsilon}$ for all $n \geq n_0$ and all $x \in X$. This implies $|f_n(x) - f(x)| < \epsilon$ for all $n \geq n_0$ and all $x \in X$, which means that $\{f_n\}$ converges uniformly to f.

Problem 9.12. *Let $f, f_1, f_2, \ldots$ be real-valued functions defined on a compact metric space (X, d) such that $x_n \to x$ in X implies $f_n(x_n) \to f(x)$ in $\mathbf{R}$. If f is continuous, then show that the sequence of functions $\{f_n\}$ converges uniformly to f.*

Solution. Assume that the functions $f, f_1, f_2, \ldots$ satisfy the stated properties and that the function $f: X \longrightarrow \mathbf{R}$ is continuous. Also, assume by way of

contradiction that the sequence $\{f_n\}$ does not converge uniformly to f. Then an easy argument shows (how?) that there exist $\varepsilon > 0$, a subsequence $\{g_n\}$ of $\{f_n\}$, and a sequence $\{x_n\}$ of X such that

$$\left|g_n(x_n) - f(x_n)\right| \geq \varepsilon \quad \text{for each } n. \tag{$\star$}$$

Since X is compact, the sequence $\{x_n\}$ has a convergent subsequence in X, say $x_{k_n} \longrightarrow x$. By the continuity of f, we see that $f(x_{k_n}) \longrightarrow f(x)$. Also, from our hypothesis, it follows that $g_{k_n}(x_{k_n}) \longrightarrow f(x)$, and so

$$\left|g_{k_n}(x_{k_n}) - f(x_{k_n})\right| \longrightarrow \left|f(x) - f(x)\right| = 0,$$

contrary to ($\star$). Therefore, the sequence $\{f_n\}$ converges uniformly to f.

Problem 9.13. *For a sequence $\{f_n\}$ of real-valued functions defined on a topological space X that converges uniformly to a real-valued function f on X establish the following.*

a. *If $x_n \to x$ and f is continuous at x, then $f_n(x_n) \to f(x)$.*
b. *If each f_n is continuous at some point $x_0 \in X$, then f is also continuous at the point x_0 and*

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = f(x_0).$$

Solution. (a) Assume f is continuous at x, $x_n \to x$ and let $\varepsilon > 0$. Choose some k with $|f_n(y) - f(y)| < \varepsilon$ for all $n > k$ and all $y \in X$. By the continuity of f at x, there exists some $m > k$ with $|f(x_n) - f(x)| < \varepsilon$ for all $n > m$. Thus,

$$\left|f_n(x_n) - f(x)\right| \leq \left|f_n(x_n) - f(x_n)\right| + \left|f(x_n) - f(x)\right| < 2\varepsilon$$

holds for all $n > m$, so that $\lim f_n(x_n) = f(x)$.

(b) Assume that each f_n is continuous at $x_0 \in X$ and let $\epsilon > 0$. Since $\{f_n\}$ converges uniformly to f on X, there exists some k satisfying $|f_k(x) - f(x)| < \epsilon$ for all $x \in X$. Now, the continuity of f_k at x_0 guarantees the existence of a neighborhood V of x_0 such that $|f_k(x) - f_k(x_0)| < \epsilon$ for all $x \in V$. Then

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(x_0)| + |f_k(x_0) - f(x_0)| \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon \end{aligned}$$

holds for all $x \in V$, which shows that f is continuous at x_0. For the equality, note that

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{x \to x_0} f(x) = f(x_0),$$

while

$$\lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} f_n(x_0) = f(x_0).$$

Problem 9.14. *Let $f_n\colon [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for $x \in [0, 1]$. Show that $\{f_n\}$ converges pointwise and find its limit function. Is the convergence uniform?*

Solution. Clearly,

$$f_n(x) \longrightarrow f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1; \\ 1, & \text{if } x = 1. \end{cases}$$

Since f is not continuous, the convergence cannot be uniform; see Theorem 9.2.

Problem 9.15. *Let $g\colon [0, 1] \to \mathbb{R}$ be a continuous function with $g(1) = 0$. Show that the sequence of functions $\{f_n\}$ defined by $f_n(x) = x^n g(x)$ for $x \in [0, 1]$, converges uniformly to the constant zero function.*

Solution. Let $\varepsilon > 0$. Choose some $0 < \delta < 1$ with $|g(x)| < \varepsilon$ whenever $\delta < x \le 1$. Now, pick some $M > 0$ with $|g(x)| \le M$ for all $x \in [0, 1]$, and then select some k with $M\delta^n < \varepsilon$ whenever $n > k$. Thus, for each $n > k$ we have $|x^n g(x)| \le M\delta^n < \varepsilon$ for $0 \le x \le \delta$ and $|x^n g(x)| \le |g(x)| < \varepsilon$ for all $\delta < x \le 1$. That is, the sequence $\{f_n\}$ converges uniformly to the constant zero function.

Problem 9.16. *Let $\{f_n\}$ be a sequence of continuous real-valued functions defined on $[a, b]$, and let $\{a_n\}$ and $\{b_n\}$ be two sequences of $[a, b]$ such that $\lim a_n = a$ and $\lim b_n = b$. If $\{f_n\}$ converges uniformly to f on $[a, b]$, then show that*

$$\lim_{n \to \infty} \int_{a_n}^{b_n} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Solution. Let $\varepsilon > 0$. Pick some k such that for all $n > k$ we have:

1. $a_n - a < \varepsilon$ and $b - b_n < \varepsilon$; and
2. $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$.

Also, since f is continuous (Theorem 9.2), there exists some $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in [a, b]$. Thus,

$$\begin{aligned}
&\left|\int_{a_n}^{b_n} f_n(x)\,dx - \int_a^b f(x)\,dx\right| \\
&= \left|\int_{a_n}^{b_n} f_n(x)\,dx - \int_{a_n}^{b_n} f(x)\,dx - \int_a^{a_n} f(x)\,dx - \int_{b_n}^{b} f(x)\,dx\right| \\
&\leq \int_a^b \big|f_n(x) - f(x)\big|\,dx + \int_a^{a_n} \big|f(x)\big|\,dx + \int_{b_n}^{b} |f(x)|\,dx \\
&\leq \varepsilon(b-a) + M(a_n - a) + M(b - b_n) < \varepsilon(2M + b - a)
\end{aligned}$$

holds for all $n > k$, and our conclusion follows.

Problem 9.17. *Let $\{f_n\}$ be a sequence of continuous real-valued functions on a metric space X such that $\{f_n\}$ converges uniformly to some function f on every compact subset of X. Show that f is a continuous function.*

Solution. Let $x_n \longrightarrow x$ in X. Put $K = \{x_1, x_2, \ldots\} \cup \{x\}$, and note that K is a compact set—every open cover of K can be reduced to a finite cover. Since $\{f_n\}$ is a sequence of continuous functions that converges uniformly to f on K, it follows from Theorem 9.2 that f is continuous on K. Since $x_n \longrightarrow x$ holds in K, we get $f(x_n) \longrightarrow f(x)$. That is, f is a continuous function.

Problem 9.18. *Let $\{f_n\}$ and $\{g_n\}$ be two uniformly bounded sequences of real-valued functions on a set X. If both $\{f_n\}$ and $\{g_n\}$ converge uniformly on X, then show that $\{f_n g_n\}$ also converges uniformly on X.*

Solution. Assume that $\{f_n\}$ and $\{g_n\}$ converge uniformly to f and g, respectively. Let $\varepsilon > 0$. Choose some k with $|f_n(x) - f(x)| < \varepsilon$ and $|g_n(x) - g(x)| < \varepsilon$ for all $n > k$ and all $x \in X$. Also, pick some $M > 0$ so that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ hold for all n and all x. Now, note that

$$\begin{aligned}
&\big|f_n(x)g_n(x) - f(x)g(x)\big| \\
&\leq \big|f_n(x)\big| \cdot \big|g_n(x) - g(x)\big| + \big|g(x)\big| \cdot \big|f_n(x) - f(x)\big| < 2M\varepsilon
\end{aligned}$$

holds for all $n > k$ and all $x \in X$.

Problem 9.19. *Suppose that $\{f_n\}$ is a sequence of monotone real-valued functions defined on $[a, b]$ and not necessarily all increasing or decreasing. Show*

that if $\{f_n\}$ *converges pointwise to a continuous function* f *on* $[a, b]$, *then* $\{f_n\}$ *converges uniformly to* f *on* $[a, b]$.

Solution. Let $\varepsilon > 0$. Since f is uniformly continuous (Theorem 7.7), there exists some $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ holds whenever $|x - y| < \delta$. Fix a finite number of points $a = x_0 < x_1 < \cdots < x_k = b$ with $x_i - x_{i-1} < \delta$ for $1 \le i \le k$, and then pick some m such that $|f_n(x_i) - f(x_i)| < \varepsilon$ holds for each $0 \le i \le k$ and all $n > m$.

Now, let $n > m$. Assume that f_n is decreasing. If $x \in [a, b]$, then $x_{i-1} \le x \le x_i$ holds for some $1 \le i \le k$, and so

$$\begin{aligned}
\big|f_n(x) - f_n(x_i)\big| &= f_n(x) - f_n(x_i) \le f_n(x_{i-1}) - f_n(x_i)\\
&= [f_n(x_{i-1}) - f(x_{i-1})] + [f(x_{i-1}) - f(x_i)] + [f(x_i) - f_n(x_i)]\\
&< \varepsilon + \varepsilon + \varepsilon < 3\varepsilon.
\end{aligned}$$

A similar inequality holds true if f_n is increasing. Therefore,

$$\begin{aligned}
\big|f_n(x) - f(x)\big| &\le \big|f_n(x) - f_n(x_i)\big| + \big|f_n(x_i) - f(x_i)\big| + \big|f(x_i) - f(x)\big|\\
&< 3\varepsilon + \varepsilon + \varepsilon = 5\varepsilon
\end{aligned}$$

holds for all $x \in [a, b]$ and all $n > m$. That is, $\{f_n\}$ converges uniformly to f.

Problem 9.20. *Let X be a topological space and let $\{f_n\}$ be a sequence of real-valued continuous functions defined on X. Suppose that there is a function $f: X \to \mathbf{R}$ such that $f(x) = \lim f_n(x)$ holds for all $x \in X$. Show that f is continuous at a point a if and only if for each $\epsilon > 0$ and each m there exist a neighborhood V of a and some $k > m$ such that $|f(x) - f_k(x)| < \epsilon$ holds for all $x \in V$.*

Solution. Assume that f is continuous at some point a. Let $\varepsilon > 0$ and an integer m be given. Pick a neighborhood U of a such that $|f(x) - f(a)| < \varepsilon$ holds for all $x \in U$. Since $\lim f_n(a) = f(a)$ holds, there exists an integer $r > m$ such that $|f(a) - f_n(a)| < \varepsilon$ holds for all $n > r$. Fix any integer $k > r$ and note that $k > m$. Since f_k is a continuous function, there exists a neighborhood W of a such that $|f_k(a) - f_k(x)| < \varepsilon$ holds for all $x \in W$. Now, note that if $x \in V = U \cap W$, then

$$\big|f(x) - f_k(x)\big| \le \big|f(x) - f(a)\big| + \big|f(a) - f_k(a)\big| + \big|f_k(a) - f_k(x)\big| < 3\varepsilon.$$

For the converse, assume that f satisfies the stated condition at the point a and let $\varepsilon > 0$. Since $f(a) = \lim f_n(a)$ holds, there exists an integer m such

that $|f(a) - f_n(a)| < \varepsilon$ holds for all $n > m$. By the hypothesis, there exist a neighborhood V of a and an integer $k > m$ such that $|f(x) - f_k(x)| < \varepsilon$ holds for all $x \in V$. By the continuity of f_k, there exists another neighborhood U of a such that $|f_k(a) - f_k(x)| < \varepsilon$ holds for all $x \in U$. Now, note that $x \in U \cap V$ implies

$$\left|f(a) - f(x)\right| \le \left|f(a) - f_k(a)\right| + \left|f_k(a) - f_k(x)\right| + \left|f_k(x) - f(x)\right| < 3\varepsilon,$$

which shows that f is continuous at the point a.

Problem 9.21. *Let $\{f_n\}$ be a uniformly bounded sequence of continuous real-valued functions on a closed interval $[a, b]$. Show that the sequence of functions $\{\phi_n\}$ defined by $\phi_n(x) = \int_a^x f_n(t)\,dt$ for each $x \in [a, b]$, contains a uniformly convergent subsequence on $[a, b]$.*

Solution. Since the sequence $\{f_n\}$ is uniformly bounded, there is some $M > 0$ such that $|f_n(x)| < M$ holds for all $x \in [a, b]$ and all n. Clearly,

$$\left|\phi_n(x)\right| = \left|\int_a^x f_n(t)\,dt\right| \le M(b-a)$$

holds for all $x \in [a, b]$ and all n. So, the sequence $\{\phi_n\}$ is uniformly bounded and we claim that it is an equicontinuous sequence.

To see this, let $\varepsilon > 0$ and put $\delta = \varepsilon/M$. Now, note that $x, y \in [a, b]$ and $|x - y| < \delta$ imply

$$\left|\phi_n(x) - \phi_n(y)\right| = \left|\int_a^x f_n(t)\,dt - \int_a^y f_n(t)\,dt\right|$$
$$\le \left|\int_x^y \left|f_n(t)\right| dt\right| \le \left|\int_x^y M\,dt\right| = M|x-y| < \varepsilon.$$

Thus, the set $A = \{\phi_1, \phi_2, , \ldots\}$ is equicontinuous. If $\overline{A}$ denotes the (uniform) closure of A in $C[a, b]$, then $\overline{A}$ is bounded, closed, and equicontinuous (why?). By the Ascoli–Arzelà theorem (Theorem 9.10), the set $\overline{A}$ is a compact set. Since $\{\phi_n\}$ is a sequence of $\overline{A}$, it follows that $\{\phi_n\}$ has a subsequence that converges uniformly on $[a, b]$.

Problem 9.22. *For each n let $f_n\colon \mathbf{R} \to \mathbf{R}$ be a monotone (either increasing or decreasing) function. If there exists a dense subset A of $\mathbf{R}$ such that $\lim f_n(x)$ exists in $\mathbf{R}$ for each $x \in A$, then show that $\lim f_n(x)$ exists in $\mathbf{R}$ at-most for all but countably many x.*

Solution. Assume that the functions f_n and the dense subset A of $\mathbb{R}$ satisfy the properties of the problem. Also, assume at the beginning that all but a finite number of the f_n are increasing functions.

Define the function $f: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \limsup f_n(x), \quad x \in \mathbb{R}.$$

Note that $f(x)$ is a real number for each $x \in \mathbb{R}$. Indeed, if $x \in \mathbb{R}$, then there exist $a, b \in A$ with $a < x < b$, and so $f_n(a) \le f_n(x) \le f_n(b)$ holds for all sufficiently large n. Consequently,

$$\begin{aligned} -\infty < \lim f_n(a) &= \limsup f_n(a) \\ &\le \limsup f_n(x) = f(x) \\ &\le \limsup f_n(b) = \lim f_n(b) < \infty \end{aligned}$$

Clearly, $f(x) = \lim f_n(x)$ holds for each $x \in A$. Next, note that f is an increasing function. Indeed, if $x < y$ holds, then from $f_n(x) \le f_n(y)$ for all sufficiently large n, we see that $f(x) = \limsup f_n(x) \le \limsup f_n(y) = f(y)$. By Problem 9.8, we know that f has at-most countably many discontinuities in every closed subinterval of $\mathbb{R}$. Hence, f has at-most countably many discontinuities (why?). Now, we claim that

$$\lim f_n(x) = f(x)$$

holds at every point of continuity of f. To see this, let x_0 be a point of continuity of f and let $\varepsilon > 0$. Pick some $\delta > 0$ such that $x_0 - \delta < x < x_0 + \delta$ implies $|f(x) - f(x_0)| < \varepsilon$, and then choose $a, b \in A$ with $x_0 - \delta < a < x_0 < b < x_0 + \delta$. Also, pick some n_0 such that for each $n \ge n_0$ the function f_n is increasing and satisfies

$$\left|f_n(b) - f(b)\right| < \varepsilon \quad \text{and} \quad \left|f_n(a) - f(a)\right| < \varepsilon.$$

Now, note that for $n \ge n_0$, we have

$$\begin{aligned} f(x_0) - f_n(x_0) &\le f(x_0) - f_n(a) \\ &= [f(x_0) - f(a)] + [f(a) - f_n(a)] < \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

and

$$\begin{aligned} f_n(x_0) - f(x_0) &\le f_n(b) - f(x_0) \\ &= [f_n(b) - f(b)] + [f(b) - f(x_0)] < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Thus, $|f_n(x_0) - f(x_0)| < 2\varepsilon$ holds for all $n \geq n_0$, proving that $\lim f_n(x_0) = f(x_0)$.

For the general case, assume that there are infinitely many increasing and infinitely many decreasing f_n. Split the sequence $\{f_n\}$ into two subsequences $\{g_n\}$ and $\{h_n\}$ such that each g_n is increasing and each h_n is decreasing. Put

$$g(x) = \limsup g_n(x) \text{ and } h(x) = \liminf h_n(x) = -\limsup[-h_n(x)],$$

and note that $g(a) = h(a)$ holds for each $a \in A$. By the above conclusion, g and h are continuous except possibly at the points of an at-most countable subset C of $\mathbf{R}$, and for each point $x \notin C$ we have

$$\lim g_n(x) = g(x) \quad \text{and} \quad \lim h_n(x) = h(x).$$

Now, let $c \notin C$ and fix $\varepsilon > 0$. Pick some $\delta > 0$ such that $|x - c| < \delta$ implies $|g(x) - g(c)| < \varepsilon$ and $|h(x) - h(c)| < \varepsilon$. Pick $a \in A$ with $|a - c| < \delta$ and note that from $g(a) = h(a)$, it follows that

$$\left|g(c) - h(c)\right| \leq \left|g(c) - g(a)\right| + \left|h(a) - h(c)\right| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $g(c) = h(c)$ holds for each $c \notin C$. This implies (how?) that $\lim f_n(c)$ exists in $\mathbf{R}$ for each $c \notin C$.

Problem 9.23. *Consider a continuous function $f\colon [0, \infty) \to \mathbf{R}$. For each n define the continuous function $f_n\colon [0, \infty) \to \mathbf{R}$ by $f_n(x) = f(x^n)$. Show that the set of continuous functions $\{f_1, f_2, \ldots\}$ is equicontinuous at $x = 1$ if and only if f is a constant function.*

Solution. Let $f \in C[0, \infty)$, let $f_n\colon [0, \infty) \longrightarrow \mathbf{R}$ be defined by $f_n(x) = f(x^n)$, and let $E = \{f_1, f_2, \ldots\}$. If f is a constant function, then it should be clear that the set E is equicontinuous at $x = 1$.

For the converse, assume that the set E is equicontinuous at $x = 1$. Fix $a > 0$ and let $\varepsilon > 0$. The equicontinuity of E at $x = 1$ guarantees the existence of some $0 < \delta < 1$ such that $|x - 1| < \delta$ implies $|f_n(x) - f_n(1)| < \varepsilon$ for each n. From $\lim \sqrt[n]{a} = 1$ (why?), we see that there exists some n_0 such that $|\sqrt[n]{a} - 1| < \delta$ holds for each $n \geq n_0$. Thus, if $n \geq n_0$, then we have

$$|f(a) - f(1)| = \left|f\left((\sqrt[n]{a}\,)^n\right) - f(1^n)\right| = |f_n(\sqrt[n]{a}\,) - f_n(1)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $f(a) = f(1)$ holds for each $a > 0$. By continuity, we see that $f(a) = f(1)$ for each $a \geq 0$, and so f is a constant function.

Problem 9.24. *Let (X, d) be a compact metric space and let $\mathcal{A}$ be an equicontinuous subset of $C(X)$. Show that $\mathcal{A}$ is uniformly equicontinuous, i.e., show that for each $\epsilon > 0$ there exists some $\delta > 0$ such that $x, y \in X$ and $d(x, y) < \delta$ imply $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{A}$.*

Solution. Let (X, d) be a compact metric space, let $\mathcal{A}$ be an equicontinuous subset of $C(X)$, and let $\varepsilon > 0$. For each $x \in X$ there exists (by the equicontinuity of $\mathcal{A}$) some $\delta_x > 0$ such that $d(x, y) < \delta_x$ implies $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{A}$. From $X = \bigcup_{x\in X} B(x, \frac{\delta_x}{2})$ and the compactness of X, we see that there exist $x_1, \dots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} B(x_i, \frac{\delta_{x_i}}{2})$.

Let $\delta = \frac{1}{2}\min\{\delta_{x_1}, \dots, \delta_{x_n}\} > 0$ and let $x, y \in X$ satisfy $d(x, y) < \delta$. Now, pick some $1 \le i \le n$ with $d(x, x_i) < \frac{\delta_{x_i}}{2}$ and observe that $|f(x) - f(x_i)| < \varepsilon$ for all $f \in \mathcal{A}$. In addition, from

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i},$$

we see that $|f(y) - f(x_i)| < \varepsilon$ holds for all $f \in \mathcal{A}$. Therefore, from the above, if $d(x, y) < \delta$, then

$$\big|f(x) - f(y)\big| \le \big|f(x) - f(x_i)\big| + \big|f(x_i) - f(y)\big| < \varepsilon + \varepsilon = 2\varepsilon$$

holds for all $f \in \mathcal{A}$. That is, $\mathcal{A}$ is a uniformly equicontinuous subset of $C(X)$.

Problem 9.25. *Let X be a connected topological space (see Problem 8.22 of Section 8 for the definition) and let $\mathcal{A}$ be an equicontinuous subset of $C(X)$. If for some $x_0 \in X$, the set of real numbers $\{f(x_0)\colon f \in \mathcal{A}\}$ is bounded, then show that $\{f(x)\colon f \in \mathcal{A}\}$ is also bounded for each $x \in X$.*

Solution. Let X be a connected topological space, let $\mathcal{A}$ be an equicontinuous subset of $C(X)$, and let $x_0 \in X$ be a point such that the collection of real numbers $\{f(x_0)\colon f \in \mathcal{A}\}$ is bounded. Consider the set

$$E = \big\{x \in X\colon \text{The set } \{f(x)\colon f \in \mathcal{A}\} \text{ is bounded}\big\}.$$

Since $x_0 \in E$, we see that E is nonempty. We claim that E is both open and closed. If this is the case, then by the connectedness of X we must have $E = X$, and the desired conclusion follows.

We shall show first that E is a closed set. To this end, let $y \in \overline{E}$. By the equicontinuity of $\mathcal{A}$, there exists a neighborhood V of y such that

$$|f(x) - f(y)| < 1$$

holds for all $x \in V$ and all $f \in \mathcal{A}$. From $y \in \overline{E}$, we see that $V \cap E \neq \emptyset$. Fix some $z \in V \cap E$, and then pick some $M > 0$ such that $|f(z)| \le M$ holds for

each $f \in \mathcal{A}$. In particular, we have

$$|f(y)| \leq |f(y) - f(z)| + |f(z)| < 1 + M$$

for all $f \in \mathcal{A}$. This means that $y \in E$, and so $\overline{E} = E$, i.e., E is a closed set.

Next, we shall establish that E is an open set. To this end, let $y \in E$. Pick some $C > 0$ such that $|f(y)| \leq C$ holds for each $f \in \mathcal{A}$. By the equicontinuity of $\mathcal{A}$, there exists a neighborhood W of y such that $|f(x) - f(y)| < 1$ holds for each $x \in W$ and all $f \in \mathcal{A}$. In particular, if $x \in W$, then

$$|f(x)| \leq |f(x) - f(y)| + |f(y)| < 1 + C$$

holds for all $f \in \mathcal{A}$, and so $x \in E$. That is, $W \subseteq E$ holds, which shows that y is an interior point of E. Therefore, E is also an open set.

Problem 9.26. *Let $\{f_n\}$ be an equicontinuous sequence in $C(X)$, where X is not necessarily compact. If for some function $f: X \to \mathbb{R}$ we have* $\lim f_n(x) = f(x)$ *for each $x \in X$, then show that $f \in C(X)$.*

Solution. Let $x \in X$ and let $\varepsilon > 0$. Since $\{f_n\}$ is an equicontinuous sequence, there exists a neighborhood V of the point x such that $|f_n(y) - f_n(x)| < \varepsilon$ holds for all n and each $y \in V$.

Now, let $y \in V$. Pick some k with $|f_k(x) - f(x)| < \varepsilon$ and $|f_k(y) - f(y)| < \varepsilon$, and note that

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| < 3\varepsilon.$$

That is, f is continuous at the arbitrary point x.

Problem 9.27. *Let X be a compact topological space, and let $\{f_n\}$ be an equicontinuous sequence of $C(X)$. Assume that there exists some $f \in C(X)$ and some dense subset A of X such that* $\lim f_n(x) = f(x)$ *holds for each $x \in A$. Then show that $\{f_n\}$ converges uniformly to f.*

Solution. Let $\varepsilon > 0$. By the equicontinuity of $\{f_n\}$ and the continuity of f, for each $x \in X$, there exists some neighborhood V_x of x such that

1. $|f_n(y) - f_n(x)| < \varepsilon$ holds for all $y \in V_x$ and all n; and
2. $|f(y) - f(x)| < \varepsilon$ holds for all $y \in V_x$.

By the compactness of X, there exist $x_1, \ldots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} V_{x_i}$.

Now, let $y \in V_{x_i}$. Choose some $x \in A \cap V_{x_i}$, and then pick some m_i with $|f_n(x) - f(x)| < \varepsilon$ for all $n > m_i$. Clearly, $|f(x) - f(y)| < 2\varepsilon$. Thus,

$$\begin{aligned}
&\big|f_n(y) - f(y)\big| \\
&\le \big|f_n(y) - f_n(x_i)\big| + \big|f_n(x_i) - f_n(x)\big| + \big|f_n(x) - f(x)\big| + \big|f(x) - f(y)\big| \\
&< 5\varepsilon
\end{aligned}$$

holds for all $y \in V_{x_i}$ and all $n > m_i$.

Finally, put $m = \max\{m_i\colon 1 \le i \le k\}$, and note that $|f_n(y) - f(y)| < 5\varepsilon$ for all $y \in X$ and all $n > m$.

Problem 9.28. *Show that for any fixed integer $n > 1$ the set of functions f in $C[0, 1]$ such that there is some $x \in [0, 1 - \frac{1}{n}]$ for which*

$$|f(x+h) - f(x)| \le nh \quad \textit{whenever} \quad 0 < h < \tfrac{1}{n},$$

is nowhere dense in $C[0, 1]$ (with the uniform metric).

Use the above conclusion and Baire's theorem to prove that there exists a continuous real-valued function defined on $[0, 1]$ that is not differentiable at any point of $[0, 1]$.

Solution. Let $D(f, g) = \|f - g\|_\infty = \sup\big\{|f(x) - g(x)|\colon x \in [0, 1]\big\}$. For $n \ge 2$ define

$$A_n = \Big\{f \in C[0, 1]\colon \exists\ x \in [0, 1 - \tfrac{1}{n}] \text{ with } \big|f(x+h) - f(x)\big| \le nh \\ \text{whenever } 0 < h < \tfrac{1}{n}\Big\}.$$

We claim that:

1. Each A_n is closed; and
2. Each A_n is nowhere dense in $C[0, 1]$ (i.e., $(A_n)^{\circ} = \emptyset$).

To see that each A_n is closed, let $\{f_k\} \subseteq A_n$ satisfy $\lim D(f_k, f) = 0$ (i.e., $\{f_k\}$ converges uniformly to f on $[0, 1]$). For each k choose some $x_k \in [0, 1 - \frac{1}{n}]$ with $|f_k(x_k + h) - f_k(x_k)| \le nh$ for all $0 < h < \frac{1}{n}$. Since $[0, 1 - \frac{1}{n}]$ is compact, there exists a subsequence of $\{x_k\}$ that converges to some $x \in [0, 1 - \frac{1}{n}]$. We can assume that $\lim x_k = x$. By Problem 9.13, $\lim f_k(x_k + h) = f(x + h)$ and $\lim f_k(x_k) = f(x)$, and so $|f(x + h) - f(x)| \le nh$ holds for all $0 < h < \frac{1}{n}$. Thus, $f \in A_n$, and hence, A_n is a closed subset of $C[0, 1]$.

Now, let $f \in A_n$ and let $\varepsilon > 0$. Consider the function $g \in C[0, 1]$ whose graph is shown in Figure 2.1. Note that for each $x \in [0, 1)$ we have $|g(x + h) - g(x)| = 3nh$ for all sufficiently small $h > 0$. Put $f_1 = f + g$, and note that

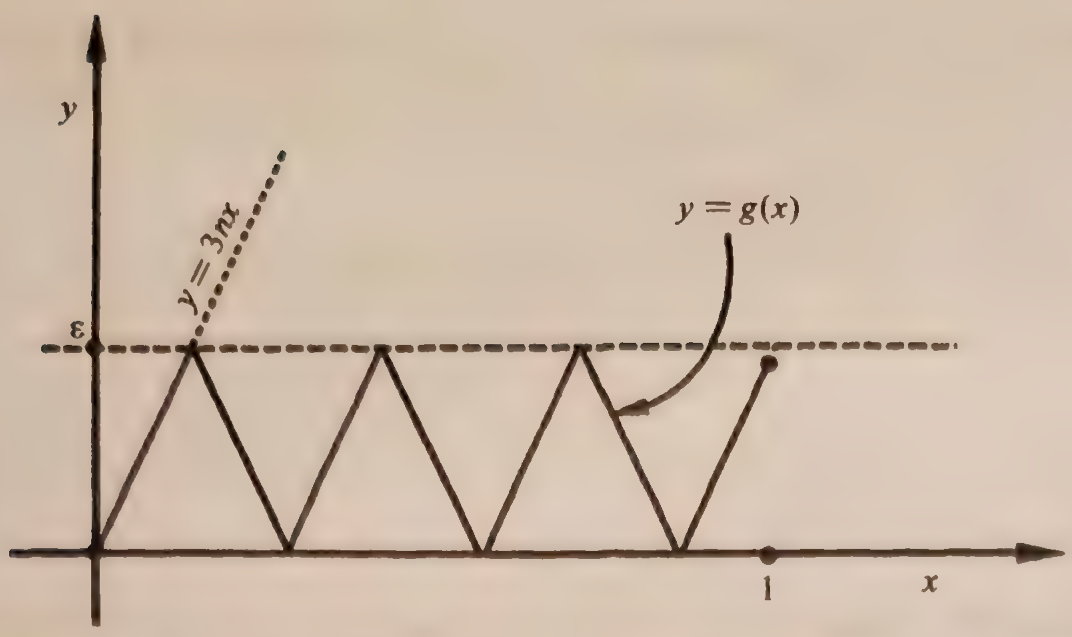

FIGURE 2.1. The Construction of a Nowhere Differentiable Function

$D(f, f_1) = \|g\|_\infty = \varepsilon$. On the other hand, if $x \in [0, 1)$ is fixed, then for all sufficiently small $h > 0$ we have

$$\begin{aligned} nh < 2nh = 3nh - nh &\le \big|g(x+h) - g(x)\big| - \big|f(x+h) - f(x)\big| \\ &\le \big|g(x+h) - g(x) - [f(x) - f(x+h)]\big| = \big|f_1(x+h) - f_1(x)\big|. \end{aligned}$$

Thus, $f_1 \notin A_n$, and so $B(f, 2\varepsilon) \not\subseteq A_n$ for all $\varepsilon > 0$. This shows that $(A_n)^\circ = \emptyset$.

Now, for each $n \ge 2$ let

$$B_n = \Big\{ f \in C[0,1]\colon \exists\, x \in [\tfrac{1}{n}, 1] \text{ with } |f(x-h) - f(x)| \le nh \text{ whenever } 0 < h < \tfrac{1}{n} \Big\}.$$

By the same arguments, each B_n is closed and nowhere dense. Consequently, from Baire's Theorem 6.17, we have

$$C[0,1] \ne \Big(\bigcup_{n=2}^{\infty} A_n\Big) \cup \Big(\bigcup_{n=2}^{\infty} B_n\Big).$$

In particular, note that every $f \in C[0,1] \setminus \big(\bigcup_{n=2}^{\infty} A_n\big) \cup \big(\bigcup_{n=2}^{\infty} B_n\big)$ does not have any one-sided derivative at any point of $[0, 1]$.

Problem 9.29. *Establish the following result regarding differentiability and uniform convergence. Let $\{f_n\}$ be a sequence of differentiable real-valued functions defined on a bounded open interval (a, b) such that:*

a. *for some $x_0 \in (a, b)$ the sequence of real numbers $\{f_n(x_0)\}$ converges in $\mathbf{R}$, and*

b. *the sequence of derivatives* $\{f_n'\}$ *converges uniformly to a function* $g: (a, b) \to \mathbb{R}$.

Then the sequence $\{f_n\}$ *converges uniformly to a function* $f: (a, b) \to \mathbb{R}$ *which is differentiable at* x_0 *and satisfies* $f'(x_0) = g(x_0)$.

Solution. First, we shall show that $\{f_n\}$ is a uniformly Cauchy sequence. To this end, let $\epsilon > 0$ and pick some $M > 0$ such that $|x - x_0| \le M$ for each $x \in (a, b)$. Next, choose some k such that

$$\left|f_n'(x) - f_m'(x)\right| < \epsilon \quad \text{for all } m, n \ge k \text{ and all } x \in (a, b) \qquad (\star)$$

and

$$\left|f_n(x_0) - f_m(x_0)\right| < \epsilon \quad \text{for all } m, n \ge k. \qquad (\star\star)$$

Using the Mean Value Theorem, $(\star)$ and $(\star\star)$, we see that for each $x \in (a, b)$ and each pair $n, m \ge k$ there exists some $t \in (a, b)$ such that

$$\begin{aligned}
\left|f_n(x) - f_m(x)\right| &\le \left|\left[f_n(x) - f_m(x)\right] - \left[f_n(x_0) - f_m(x_0)\right]\right| + \left|f_n(x_0) - f_m(x_0)\right| \\
&= \left|f_n'(t) - f_m'(t)\right| \cdot |x - x_0| + \left|f_n(x_0) - f_m(x_0)\right| \\
&\le M\epsilon + \epsilon = (1 + M)\epsilon.
\end{aligned}$$

This shows that $\{f_n\}$ is a uniformly Cauchy sequence, and hence, $\{f_n\}$ converges uniformly to a function $f: (a, b) \to \mathbb{R}$.

Next, for each n we consider the continuous function $\phi_n: (a, b) \to \mathbb{R}$ defined by $\phi_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0}$ if $x \ne x_0$ and $\phi_n(x_0) = f_n'(x_0)$. Using the Mean Value Theorem and $(\star)$, we see that for each $x \in (a, b)$ there exists some $c_x \in (a, b)$ such that

$$\left|\phi_n(x) - \phi_m(x)\right| = \left|\frac{[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]}{x - x_0}\right| = \left|f_n'(c_x) - f_m'(c_x)\right| < \epsilon,$$

for all $n, m \ge k$. This shows that $\{\phi_n\}$ is a uniformly Cauchy sequence, and hence, it converges uniformly to the function $\phi: (a, b) \to \mathbb{R}$ defined by $\phi(x) = \frac{f(x) - f(x_0)}{x - x_0}$ if $x \ne x_0$ and $\phi(x_0) = g(x_0)$.

Finally, from Problem 9.13, we obtain

$$\begin{aligned}
g(x_0) &= \lim_{n\to\infty} f_n'(x_0) = \lim_{n\to\infty} \lim_{x\to x_0} \phi_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} \phi_n(x) \\
&= \lim_{x\to x_0} \lim_{n\to\infty} \frac{f_n(x) - f(x)}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\end{aligned}$$

This shows that f is differentiable at x_0 and that $f'(x_0) = g(x_0)$.

10. SEPARATION PROPERTIES OF CONTINUOUS FUNCTIONS

Problem 10.1. *Let (X, d) be a metric space and let A be a nonempty subset of X. The* **distance function** *of A is the function $d(\cdot, A)\colon X \to \mathbb{R}$ defined by*

$$d(x, A) = \inf\{d(x, a)\colon a \in A\}.$$

Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

Solution. Clearly, $d(x, A) \geq 0$ for each $x \in X$. Assume that $x \in \overline{A}$ and let $\epsilon > 0$. Then $B(x, \epsilon) \cap A \neq \emptyset$, and so there exists some $y \in A$ such that $d(x, y) < \epsilon$. From the definition of the distance function, we see that $0 \leq d(x, A) \leq d(x, y) < \epsilon$. Since $\epsilon > 0$ is arbitrary, this implies $d(x, A) = 0$.

For the converse, assume that $d(x, A) = 0$. If $\epsilon > 0$, then it follows from $d(x, A) = \inf\{d(x, a)\colon a \in A\} < \epsilon$ that there exists some $a \in A$ with $d(x, a) < \epsilon$. Hence, $B(x, \epsilon) \cap A \neq \emptyset$ for each $\epsilon > 0$, and this implies that $x \in \overline{A}$.

Problem 10.2. *Let (X, d) be a metric space, let A and B be two nonempty disjoint closed sets and consider the function $f\colon X \to [0, 1]$ defined by $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}$. Show that:*

a. *f is a continuous function,*
b. *$f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$, and*
c. *if $d(A, B) = \inf\{d(a, b)\colon a \in A \text{ and } b \in B\} > 0$, then f is uniformly continuous.*

Solution. Let C be an arbitrary nonempty subset of X. We shall show first that the function $x \mapsto d(x, C)$ is uniformly continuous. To see this, fix $x, y \in X$. Choosing some $c \in C$, we see that

$$d(x, C) \leq d(x, c) \leq d(x, y) + d(y, c) \leq d(x, y) + d(y, C),$$

or $d(x, C) - d(y, C) \leq d(x, y)$. Exchanging the roles of x and y in the last inequality, we get $d(y, C) - d(x, C) \leq d(x, y)$. Therefore,

$$\left|d(x, C) - d(y, C)\right| \leq d(x, y),$$

and the uniform continuity of $x \mapsto d(x, C)$ follows.

(a) Observe that since A and B are disjoint closed sets, it follows from the Problem 10.1 that $d(x, A) + d(x, B) > 0$ for each $x \in X$. This, in connection with the (uniform) continuity of the functions $d(\cdot, A)$ and $d(\cdot, B)$, guarantees that f is a continuous function.

(b) Note that $f(x) = 0$ if and only if $d(x, A) = 0$. Now, by Problem 10.1, we have $d(x, A) = 0$ if and only if $x \in \overline{A} = A$. In other words, we have $f(x) = 0$ if and only if $x \in A$. This means $f^{-1}(\{0\}) = A$.

Similarly, notice that $f(x) = \frac{d(x,A)}{d(x,A)+d(x,B)} = 1$ if and only if $d(x, B) = 0$. As above, this shows that $f^{-1}(\{1\}) = B$.

(c) Fix some $\epsilon > 0$ such that $d(u, v) \geq \epsilon$ for all $u \in A$ and $v \in B$. If $a \in A$ and $b \in B$ are arbitrary, then for each $z \in X$ we have

$$\epsilon \leq d(a, b) \leq d(z, a) + d(z, b) \leq d(z, A) + d(z, B).$$

Now, if $x, y \in X$, then the inequalities

$$\begin{aligned}
|f(x) - f(y)| &= \left| \frac{d(x, A)}{d(x, A) + d(x, B)} - \frac{d(y, A)}{d(y, A) + d(y, B)} \right| \\
&= \frac{|[d(y, A) + d(y, B)]d(x, A) - [d(x, A) + d(x, B)]d(y, A)|}{[d(x, A) + d(x, B)][d(y, A) + d(y, B)]} \\
&= \frac{|[d(x, A) - d(y, A)]d(x, B) + [d(y, B) - d(x, B)]d(x, A)|}{[d(x, A) + d(x, B)][d(y, A) + d(y, B)]} \\
&\leq \frac{[d(x, B) + d(x, A)]d(x, y)}{[d(x, A) + d(x, B)][d(y, A) + d(y, B)]} \\
&\leq \frac{d(x, y)}{\epsilon}
\end{aligned}$$

guarantee that f is uniformly continuous.

Problem 10.3. *Let A and B be two nonempty subsets of a metric space X such that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Show that there exist two open disjoint set U and V such that $A \subseteq U$ and $B \subseteq V$.*

Solution. From the solution of Problem 10.2, we know that for each nonempty subset C of X the function $x \mapsto d(x, C)$ is (uniformly) continuous. Now, consider the function $f: X \to \mathbb{R}$ defined by

$$f(x) = d(x, A) - d(x, B).$$

By the above, f is a continuous function. From $A \cap \overline{B} = \emptyset$ and Problem 10.1, we see that $f(x) = -d(x, B) < 0$ holds for each $x \in A$. Similarly, $f(x) > 0$ holds for each $x \in B$. Consequently, the two disjoint open sets $U = f^{-1}\big((-\infty, 0)\big)$ and $V = f^{-1}\big((0, \infty)\big)$ satisfy $A \subseteq U$ and $B \subseteq V$.

Problem 10.4. *Show that a closed set of a normal space is itself a normal space.*

Solution. Let C be a closed subset of a normal space X. We consider C equipped with the topology induced by X. Now, assume that A and B are two disjoint closed subsets of C. Since C is closed, it is easy to see that A and B are also closed subsets

of X. Pick two open subsets V_1 and W_1 of X satisfying $A \subseteq V_1$, $B \subseteq W_1$ and $V_1 \cap W_1 = \emptyset$. Now, if $V = C \cap V_1$ and $W = C \cap W_1$, then V and W are two disjoint open subsets of C satisfying $A \subseteq V$ and $B \subseteq W$. This shows that C equipped with the relative topology is a normal space.

Problem 10.5. *Let X be a normal space and let A and B be two disjoint closed subsets of X. Show that there exist open sets V and W such that $A \subseteq V$, $B \subseteq W$ and $\overline{V} \cap \overline{W} = \emptyset$.*

Solution. Assume that A and B are two disjoint closed subsets of a normal space X. Pick two disjoint open sets V and W_1 satisfying $A \subseteq V$ and $B \subseteq W_1$. We claim that $\overline{V} \cap W_1 = \emptyset$. Indeed, if $x \in \overline{V} \cap W_1$, then on one hand W_1 is a neighborhood of x, and on the other hand, x belongs to the closure of V, which imply $W_1 \cap V \neq \emptyset$, a contradiction.

Now, since $\overline{V} \cap B = \emptyset$ and X is normal, there exist two disjoint open sets V_1 and W such that $\overline{V} \subseteq V_1$ and $B \subseteq W$. As before, $V_1 \cap \overline{W} = \emptyset$, and clearly the open sets V and W satisfy the desired properties.

Alternatively: If a continuous function $f\colon X \to [0,1]$ satisfies $A \subseteq f^{-1}(\{0\})$ and $B \subseteq f^{-1}(\{1\})$, then the open sets $V = f^{-1}\big([0,\frac{1}{2})\big)$ and $W = f^{-1}\big((\frac{3}{4},1]\big)$ satisfy $A \subseteq \overline{V}$, $B \subseteq \overline{W}$, and $\overline{V} \cap \overline{W} = \emptyset$.

Problem 10.6. *Show that a topological space is normal if and only if for each closed set A and each open set V with $A \subseteq V$, there exists an open set W such that $A \subseteq W \subseteq \overline{W} \subseteq V$.*

Solution. Let X be a topological space. Assume first that X is a normal space and let a closed set A and an open set V satisfy $A \subseteq V$. Then $A \cap V^c = \emptyset$ and V^c is a closed set. Pick two disjoint open sets W and U such that $A \subseteq W$ and $V^c \subseteq U$. In particular, $\overline{W} \cap U = \emptyset$. This implies $\overline{W} \cap V^c = \emptyset$, and so $\overline{W} \subseteq V$.

For the converse, assume that the property is satisfied and let A and B be two disjoint nonempty closed sets. If $V = B^c$, then V is an open set such that $A \subseteq V$. By our hypothesis, there exists an open set W such that $A \subseteq W \subseteq \overline{W} \subseteq V = B^c$. If $U = \overline{W}^c$, then U is an open set disjoint from W and satisfies $B \subseteq U$. This shows that X is a normal space.

Problem 10.7. *For a closed subset A of a normal topological space X, establish the following:*

a. *There exists a continuous function $f\colon X \to [0,1]$ satisfying $f^{-1}(\{0\}) = A$ if and only if A is a G_δ-set.*

b. *If A is a G_δ-set and B is another closed set satisfying $A \cap B = \emptyset$, then there exists a continuous function $g: X \to [0, 1]$ such that $g^{-1}(\{0\}) = A$ and $g(b) = 1$ for each $b \in B$.*

Solution. Let A be a closed subset of a normal topological space X.

(a) If there exists a continuous function $f: X \to [0, 1]$ such that $f^{-1}(\{0\}) = A$, then the identity

$$A = f^{-1}(\{0\}) = f^{-1}\Big(\bigcap_{n=1}^{\infty}[0, \tfrac{1}{n})\Big) = \bigcap_{n=1}^{\infty} f^{-1}([0, \tfrac{1}{n}))$$

shows that A is a G_δ-set.

For the converse, assume that A is a G_δ-set. Pick a sequence $\{V_n\}$ of open sets such that $A = \bigcap_{n=1}^{\infty} V_n$. Since $A \cap V_n^c = \emptyset$, it follows from Uryson's lemma that there exists a continuous function $f_n: X \to [0, 1]$ satisfying $f_n(a) = 0$ for each $a \in A$ and $f_n(x) = 1$ for all $x \in V_n^c$. Now, consider the function $f: X \to [0, 1]$ defined by

$$f(x) = \sum_{n=1}^{\infty} \tfrac{1}{2^n} f_n(x).$$

From the Weierstrass' M-test (Theorem 9.5) and Theorem 9.2, it is easy to see that f is a continuous function, and we claim that $f^{-1}(\{0\}) = A$. Clearly, $f(x) = 0$ for each $x \in A$. Now, assume $f(x) = 0$. Then $f_n(x) = 0$ for all n, and so (in view of $f_n(v) = 1$ for each $v \in V_n^c$) we have $x \in V_n$ for each n, i.e., $x \in \bigcap_{n=1}^{\infty} V_n = A$. Therefore, $f^{-1}(\{0\}) = A$.

(b) Assume now that A is a closed G_δ-set and B is another closed set such that $A \cap B = \emptyset$. So, there exist two disjoint open set V and W such that $A \subseteq V$ and $B \subseteq W$. This implies that the sequence $\{V_n\}$ introduced in part (a) can be assumed to satisfy $V_n \subseteq V$ for each n. In particular, each f_n satisfies $f_n(b) = 1$ for each $b \in B$. Now, it is easy to see that the continuous function f constructed in the preceding part satisfies the desired property.

Problem 10.8. *Show that a compact subset A of a Hausdorff locally compact topological space is a G_δ-set if and only if there exists a continuous function $f: X \to [0, 1]$ such that $A = f^{-1}(\{0\})$.*

Solution. If $A = f^{-1}(\{0\})$, then—as in the solution of part (a) of the preceding problem—the set A is a G_δ-set. For the converse, assume that $A = \bigcap_{n=1}^{\infty} V_n$, where each V_n is an open set. By Theorem 10.8, for each n there exists a continuous

function $f_n: X \to [0, 1]$ such that $f_n(x) = 1$ for each $x \in A$ and $f_n(x) = 0$ for each $x \notin V_n$. Now, as in the the solution of part (a) of the preceding problem, notice that the function $f: X \to [0, 1]$ defined by $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x)$ satisfies the desired properties.

Problem 10.9. *A topological space X is said to be* **perfectly normal** *if for every pair of disjoint closed sets A and B, there is a continuous function $f: X \to [0, 1]$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. (Part (b) of Problem 10.2 shows that every metric space is perfectly normal.)*

Show that a Hausdorff normal topological space is perfectly normal if and only if every closed set is a G_δ-set.

Solution. Let X be a Hausdorff normal topological space. Assume first that X is perfectly normal and let A be a proper closed subset of X. If $a \in X$ satisfies $a \notin A$, then $A \cap \{a\} = \emptyset$ and $\{a\}$ is a closed set. So, there exists a continuous function $f: X \to [0, 1]$ with $f^{-1}(\{0\}) = A$. This implies (as in the solution of Problem 10.7), that A is a G_δ-set.

For the converse, assume that every closed set is a G_δ-set. Let A and B be two closed disjoint sets. By Problem 10.7 there exist two continuous functions $g, h: X \to [0, 1]$ such that:

i. $g^{-1}(\{0\}) = A$ and $g(b) = 1$ for each $b \in B$, and
ii. $h^{-1}(\{0\}) = B$ and $h(a) = 1$ for each $a \in A$.

Now, let $f = \frac{1}{2}g + \frac{1}{2}(1 - h)$, and note that $f: X \to [0, 1]$, $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.

Problem 10.10. *Show that a nonempty connected normal space is either a singleton or uncountable.*

Solution. Let X be a (nonempty) Hausdorff connected normal space. If X is not a singleton, then there exist $a, b \in X$ with $a \neq b$. Since X is Hausdorff, singletons are closed sets, and we have $\{a\} \cap \{b\} = \emptyset$. Now, pick a continuous function $f: X \to [0, 1]$ such that $f(a) = 0$ and $f(b) = 1$. The assumption that X is connected guarantees (according to Problem 6.11(g)) that $f(X)$ is an interval and so $f(X) = [0, 1]$. This easily implies that X is uncountable—in fact, it has cardinality greater than or equal to the cardinality of the continuum.

Problem 10.11. *Let X be a normal space, let C be a closed subset of X, and let I be a nonempty interval—with the possibility $I = (-\infty, \infty)$. If $f: C \to I$ is a continuous function, then show that f has a continuous extension to all of X with values in I.*

Solution. Assume that C is a closed subset of a normal space X and that $f: X \to I$ is a continuous function, where I is an interval. The interval I must be one of the following type: (a, b), $[a, b]$, $[a, b)$, $(a, b]$. So, we shall establish the continuous extension of f by steps.

STEP I: *I is either of the form $[a, b)$ or $(b, a]$.*

In this case, there exists a homeomorphism $h: I \to [0, 1)$. For instance, if $-\infty < a < b < \infty$, then $h(x) = \frac{b-x}{b-a}$ is a homeomorphism between $(a, b]$ and $[0, 1)$. Likewise, if $a \in \mathbb{R}$, then $h(x) = \frac{a-x}{1+a-x}$ defines a homeomorphism between $(-\infty, a]$ and $[0, 1)$.

Fix a homeomorphism $h: I \to [0, 1)$ and consider the continuous (composition) function $h \circ f: C \to [0, 1) \subseteq [0, 1]$. By Tietze's extension theorem, there exists a continuous function $g: X \to [0, 1]$ satisfying $g(x) = h\big(f(x)\big)$ for all $x \in C$. The continuity of g guarantees that the set $A = g^{-1}\big(\{1\}\big)$ is a closed subset of X. Also, since for each $x \in C$, we have $g(x) = h\big(f(x)\big) \in [0, 1)$, we see that $C \cap A = \emptyset$. By Uryson's lemma, there exists a continuous function $\theta: X \to [0, 1]$ such that $\theta(a) = 0$ for each $a \in A$ and $\theta(c) = 1$ for each $c \in C$.

Now, consider the function $\phi: X \to [0, 1]$ defined by $\phi(x) = \theta(x)g(x)$. We claim that $\phi(X) \subseteq [0, 1)$. To see this, let $x \in X$. If $x \in A$, then $\phi(x) = \theta(x)g(x) = 0 \cdot 1 = 0$, and if $x \notin A$, then $0 \le g(x) < 1$ and so $\phi(x) = \theta(x)g(x) < 1$ is also true.

Next, define the function $\hat{f}: X \to I$ by

$$\hat{f}(x) = (h^{-1} \circ \phi)(x) = h^{-1}\big(\theta(x)g(x)\big).$$

If $x \in C$, then $\theta(x)g(x) = g(x) = h\big(f(x)\big)$, and hence,

$$\hat{f}(x) = h^{-1}\Big(h\big(f(x)\big)\Big) = f(x).$$

This shows that $\hat{f}: X \to I$ is a continuous extension of f to all of X.

STEP II: *$I = [a, b]$ with $-\infty < a < b < \infty$.*

The function $h: [a, b] \to [0, 1]$, defined by $h(x) = \frac{x-a}{b-a}$, is a homeomorphism. By Tietze's extension theorem, there exists a continuous function $g: X \to [0, 1]$ satisfying $g(x) = (h \circ f)(x)$ for each $x \in C$. Then the continuous function $\hat{f} = h^{-1} \circ g: X \to [a, b]$ satisfies $\hat{f}(c) = f(c)$ for each $c \in C$.

STEP III: *Assume $I = (a, b)$ with $-\infty \le a < b \le \infty$.*

In this case, there exists a homeomorphism $h: (a, b) \to (-1, 1)$. (For instance, for $-\infty < a < b < \infty$ let $h(x) = \frac{2(x-a)}{b-a} - 1$ and if $(a, b) = (-\infty, \infty)$ take $h(x) = \frac{2}{\pi} \arctan x$.) Now, consider the continuous function $h \circ f: C \to (-1, 1) \subseteq [-1, 1]$ and note that by STEP II there exists a continuous function $g: X \to [-1, 1]$ satisfying $g(c) = (h \circ f)(c)$ for each $c \in C$.

Next, let $B = g^{-1}(\{-1, 1\})$. Then B is closed and $B \cap C = \emptyset$. By Uryson's lemma, there exists a continuous function $\theta: X \to [0, 1]$ satisfying $\theta(b) = 0$ for each $b \in B$ and $\theta(c) = 1$ for each $c \in C$. As before, define the continuous function $\phi: X \to [-1, 1]$ by $\phi(x) = \theta(x)g(x)$. Then it is easy to see that $\phi(X) \subseteq (-1, 1)$ and the function $\hat{f}: X \to (a, b)$, defined by $\hat{f} = h^{-1} \circ \phi$, is a continuous extension of f.

11. THE STONE–WEIERSTRASS APPROXIMATION THEOREM

Problem 11.1. *Let X be a compact topological space. For a subset L of $C(X)$, let $\overline{L}$ denote the uniform closure of L in $C(X)$. Show the following:*

a. *If L is a function space, then so is $\overline{L}$.*
b. *If L is an algebra, then so is $\overline{L}$.*

Solution. Let $f, g \in \overline{L}$. Pick two sequences $\{f_n\}$ and $\{g_n\}$ of L that converge uniformly to f and g, respectively. Also, pick some $M > 0$ so that $\|f_n\|_\infty \leq M$ and $\|g_n\|_\infty \leq M$ hold for all n.
(a) The inequality $\big||f_n| - |f|\big| \leq |f_n - f|$ shows that $\{|f_n|\}$ converges uniformly to $|f|$. Since $|f_n| \in L$ for each n, it follows that $|f| \in \overline{L}$. This implies that $\overline{L}$ is a function space.
(b) From the inequalities

$$\begin{aligned}\|f_n g_n - fg\|_\infty &\leq \|g\|_\infty \cdot \|f_n - f\|_\infty + \|f_n\|_\infty \cdot \|g_n - g\|_\infty \\ &\leq M\big(\|f_n - f\|_\infty + \|g_n - g\|_\infty\big),\end{aligned}$$

it follows that the sequence $\{f_n g_n\}$ of L converges uniformly to fg. Thus, $fg \in \overline{L}$, and so $\overline{L}$ is an algebra.

Problem 11.2. *Let L be the collection of all continuous piecewise linear functions defined on $[0, 1]$. That is, $f \in L$ if and only if $f \in C[0, 1]$ and there exists a finite number of points $0 = x_0 < x_1 < \cdots < x_n = 1$ (depending on f) such that f is linear on each interval $[x_{m-1}, x_m]$. Show that L is a function space but not an algebra. Moreover, show that L is dense in $C[0, 1]$ with respect to the uniform metric.*

Solution. The verification that L is a function space is routine. Since the function $f(x) = x$ satisfies $f \in L$ and $f^2 \notin L$, it follows that L is not an algebra of functions.

To see that L is dense, let $f \in C[0, 1]$ and let $\varepsilon > 0$. By the uniform continuity of f, there exists some $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be a finite collection of points with $x_i - x_{i-1} < \delta$

for each $1 \le i \le n$. The function g, defined on each subinterval $[x_{i-1}, x_i]$ by

$$g(t) = f(x_{i-1}) + \tfrac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}(t - x_{i-1}),$$

belongs to L and satisfies $\|f - g\|_\infty < \varepsilon$.

An alternate way of proving the denseness of L is the following: Note that $\mathbf{1} \in L$ and L separates the points of $[0, 1]$ (why?). Thus, by the Stone–Weierstrass theorem, L is dense in $C[0, 1]$.

Problem 11.3. *Show that a continuous function $f\colon (0, 1) \to \mathbb{R}$ is the uniform limit of a sequence of polynomials on $(0, 1)$ if and only if it admits a continuous extension to $[0, 1]$.*

Solution. Let $f\colon (0, 1) \longrightarrow \mathbb{R}$ be a continuous function. Assume first that f has a continuous extension to $[0, 1]$—which we denote by $\hat{f}$. Then, by Corollary 11.6, the function $\hat{f}$ is the uniform limit of a sequence of polynomials on $[0, 1]$, and consequently $f\colon (0, 1) \longrightarrow \mathbb{R}$ is likewise the uniform limit of a sequence of polynomials on $(0, 1)$.

For the converse, assume that there exists a sequence of polynomials $\{p_n\}$ that converges uniformly to f on $(0, 1)$. Let $\varepsilon > 0$ and then pick some n_0 such that $|p_n(x) - f(x)| < \varepsilon$ holds for all $x \in (0, 1)$ and all $n \ge n_0$. From the triangle inequality, we see that

$$|p_n(x) - p_m(x)| \le |p_n(x) - f(x)| + |p_m(x) - f(x)| < \varepsilon + \varepsilon = 2\varepsilon$$

for all $x \in (0, 1)$ and all $n \ge n_0$. By continuity, we infer that

$$|p_n(x) - p_m(x)| \le 2\varepsilon$$

holds for all $x \in [0, 1]$ and all $n \ge n_0$. The above show that $\{p_n\}$ is a Cauchy sequence of C[0, 1], and so (by Theorem 9.3) the sequence $\{p_n\}$ converges in C[0, 1], say to $g \in$ C[0, 1]. It follows that $f(x) = g(x)$ for all $x \in (0, 1)$, and so g is a continuous extension of $f\colon (0, 1) \longrightarrow \mathbb{R}$ to $[0, 1]$.

Problem 11.4. *If f is a continuous function on $[0, 1]$ such that $\int_0^1 x^n f(x)\,dx = 0$ for $n = 0, 1, \ldots$, then show that $f(x) = 0$ for all $x \in [0, 1]$.*

Solution. By Corollary 11.6, there exists a sequence of polynomials $\{p_n\}$ that converges uniformly to f. It easily follows that $\{p_n f\}$ also converges uniformly to f^2, and by our hypothesis we see that $\int_0^1 p_n(x)f(x)\,dx = 0$ holds for each n. Now, invoke Problem 9.16 to infer that $\int_0^1 f^2(x)\,dx = \lim \int_0^1 p_n(x)f(x)\,dx = 0$. The latter easily implies that $f(x) = 0$ holds for each $x \in [0, 1]$.

Problem 11.5. *Show that the algebra generated by the set $\{1, x^2\}$ is dense in $C[0,1]$ but fails to be dense in $C[-1,1]$.*

Solution. Since the function $f(x) = x^2$ separates the points of $[0,1]$, the algebra generated by $\{1, x^2\}$ also separates the points of $[0,1]$. Thus, by the Stone–Weierstrass, this algebra must be dense in $C[0,1]$.

To see that the algebra generated by $\{1, x^2\}$ is not dense in $C[-1,1]$, note that for every f in the closure of this algebra, we have $f(-1) = f(1)$. Thus, this algebra is not dense in $C[-1,1]$.

Problem 11.6. *Let us say that a polynomial is* **odd** *(resp.* **even**) *whenever it does not contain any monomial of even (resp. odd) degree.*

Show that a continuous function $f\colon [0,1] \to \mathbb{R}$ vanishes at zero (i.e., $f(0) = 0$) if and only if it is the uniform limit of a sequence of odd polynomials on $[0,1]$.

Solution. If f is the uniform limit of a sequence of odd polynomials, then it should be clear that f vanishes at zero. For the converse, assume that $f \in C[0,1]$ satisfies $f(0) = 0$ and let $\varepsilon > 0$. Define the function $g\colon [-1,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x), & \text{if } 0 \le x \le 1; \\ -f(-x), & \text{if } -1 \le x < 0, \end{cases}$$

and note that $g \in C[-1,1]$. By the Stone–Weierstrass theorem there exists a polynomial p such that $|g(x) - p(x)| < \varepsilon$ for each $x \in [-1,1]$.

Next, write $p = q + r$, where q is the odd polynomial consisting of the sum of all odd terms of p and r is the even polynomial consisting of the sum of all even terms (including the constant term) of p. In particular, note that $q(-x) = -q(x)$ and $r(-x) = r(x)$ hold for each x. Thus, if $0 \le x \le 1$, then

$$\big|f(x) - q(x) - r(x)\big| = \big|g(x) - p(x)\big| < \varepsilon,$$

and $g(-x) = -f(x)$ implies

$$\big|f(x) - q(x) + r(x)\big| = \big|p(-x) - g(-x)\big| < \varepsilon.$$

from which it follows that $\big|f(x) - q(x)\big| < \varepsilon$. (Here we use the elementary property: if $|a+b| < \varepsilon$ and $|a-b| < \varepsilon$, then $|a| = \big|\frac{a+b}{2} + \frac{a-b}{2}\big| \le \big|\frac{a+b}{2}\big| + \big|\frac{a-b}{2}\big| < \varepsilon$ and $|b| < \varepsilon$.) In other words, the odd polynomial q is ε-uniformly close to f on $[0,1]$, and the desired conclusion follows.

Problem 11.7. *If $f\colon[0,1]\to\mathbf{R}$ is a continuous function such that $\int_0^1 f(\sqrt[2n+1]{x})\,dx = 0$ for $n = 0, 1, 2, \ldots$, then show that $f(x) = 0$ for all $x \in [0,1]$. Does the same conclusion hold true if the interval $[0,1]$ is replaced by the interval $[-1,1]$?*

Solution. Assume that a continuous function $f \in C[0,1]$ satisfies $\int_0^1 f(\sqrt[2n+1]{x})\,dx = 0$ for each $n = 0, 1, 2, \ldots$. Then the change of variable $u = \sqrt[2n+1]{x}$ (or $x = u^{2n+1}$) yields

$$\int_0^1 f(\sqrt[2n+1]{x})\,dx = (2n+1)\int_0^1 u^{2n} f(u)\,du = 0,$$

and so $\int_0^1 x^{2n} f(x)\,dx = 0$ holds for all $n = 0, 1, 2, \ldots$. The conclusion now follows immediately from Problem 11.5.

The conclusion is not valid if we replace the interval $[0,1]$ by the interval $[-1,1]$. For instance, if $f(x) = x$ for all $x \in [-1,1]$, then note that $\int_{-1}^1 f(\sqrt[2n+1]{x})\,dx = 0$ holds for all $n = 0, 1, 2, \ldots$.

Problem 11.8. *Assume that a function $f\colon[0,\infty)\to\mathbf{R}$ is either a polynomial or else a continuous bounded function. Then show that f is identically equal to zero (i.e., show that $f = 0$) if and only if $\int_0^\infty f(x)e^{-nx}\,dx = 0$ for all $n = 1, 2, 3, \ldots$.*

Solution. Let $f\colon[0,\infty)\longrightarrow\mathbf{R}$ be a continuous bounded function. If $f = 0$, then clearly $\int_0^\infty f(x)e^{-nx}\,dx = 0$ holds for all $n = 1, 2, 3, \ldots$.

For the converse, assume that

$$\int_0^\infty f(x)e^{-nx}\,dx = 0 \quad \text{holds for all} \quad n = 1, 2, 3, \ldots. \tag{$\star$}$$

Using the change of variable $u = e^{-x}$, it follows from ($\star$) that

$$\int_0^\infty f(x)e^{-nx}\,dx = \int_{0^+}^1 f(-\ln u)u^{n-1}\,du = 0, \quad n = 1, 2, \ldots. \tag{$\star\star$}$$

In particular, $\int_{0^+}^1 g(u)u^n\,du = 0$ holds for each $n = 0, 1, \ldots$, where $g(u) = uf(-\ln u)$. Since f is bounded, note that $\lim_{u\to 0^+} g(u) = 0$ holds, and so g defines a continuous function on $[0,1]$. From ($\star\star$), we see that $\int_0^1 g(x)x^n\,dx = 0$ holds for all $n = 0, 1, 2, \ldots$. Problem 11.4 implies that $g = 0$, and consequently $f = 0$.

A closer look at the above arguments reveals that we have actually proven the following result.

- *Assume that $f:[0,\infty)\to\mathbf{R}$ is a continuous function such that*

$$\int_0^\infty f(x)e^{-nx}\,dx = 0 \quad \textit{for all} \quad n = k, k+1, k+2, \ldots,$$

where k is a positive integer. If $\lim_{u\to 0^+} u^m f(-\ln u) = 0$ for some natural number m, then the function f is identically equal to zero.

Indeed, replacing n by $n+k+m+1$ in ($\star\star$), we get

$$\int_{0^+}^1 u^{m+k} f(-\ln u)u^n\,du = 0, \quad n = 0, 1, 2, \ldots,$$

which implies (as above) that $f = 0$. The reader can verify easily that any function f that satisfies $|f(x)| \le Ce^{\alpha x}$ for some $C > 0$ and $\alpha > 0$ and all $x \ge x_0$ also satisfies $\lim_{u\to 0^+} u^m f(-\ln u) = 0$ for some natural number m. In particular, the reader should notice that every polynomial p satisfies an estimate of the form $|p(x)| \le Ce^{\alpha x}$.

One more comment regarding the above discussion is in order. Recall that if $f:[0,\infty)\to\mathbf{R}$ is a "nice" function, then the formula

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\,dt$$

is called the **Laplace transform** of f. The Laplace transform is a linear operator and plays an important role in a wide range of applications. The reader should notice that in actuality property ($\bullet$) asserts that the Laplace transform is a one-to-one operator when defined on an appropriate linear space of functions. (See also Example 30 of Chapter 5 in the text.)

Problem 11.9. *Show that a continuous bounded function $f:[1,\infty)\to\mathbf{R}$ is identically equal to zero if and only if $\int_1^\infty x^{-n} f(x)\,dx = 0$ for each $n = 8, 9, 10, \ldots$.*

Solution. The "if" part only needs verification. Therefore, assume that the function $f:[1,\infty)\longrightarrow\mathbf{R}$ satisfies $\int_1^\infty x^{-n} f(x)\,dx = 0$ for each $n = 8, 9, 10, \ldots$. Using the change of variable $u = x^{-1}$, we see that

$$\int_1^\infty x^{-n} f(x)\,dx = \int_{0^+}^1 u^{n-2} f\left(\tfrac{1}{u}\right) du = \int_{0^+}^1 u^{n-8} g(u)\,du = 0, \qquad (\star\star\star)$$

where $g(u) = u^6 f(\frac{1}{u})$. Since f is bounded, we see that $\lim_{u\to 0^+} g(u) = 0$, and so g defines a continuous function on $[0,1]$. In addition, from ($\star\star\star$), we see that

$\int_0^1 x^n g(x)\,dx = 0$ holds for each $n = 0, 1, 2, \ldots$. By Problem 11.4, it follows that $g = 0$, and consequently, $f = 0$.

Problem 11.10. *Let $\mathcal{A}$ be an algebra of continuous real-valued functions defined on a compact topological space X and separating the points of X. Show that the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $C(X)$ with respect to the uniform metric is either all of $C(X)$ or else that there exists $a \in X$ such that $\overline{\mathcal{A}} = \{f \in C(X)\colon\ f(a) = 0\}$.*

Solution. Let $\mathcal{A} \subseteq C(X)$ be an algebra, where X is compact. Now, consider the sequence of polynomials $\{P_n(x)\}$ on $[0, 1]$ defined by

$$P_1(x) = 0 \quad \text{and} \quad P_{n+1}(x) = P_n(x) + \tfrac{1}{2}\left[x - \left(P_n(x)\right)^2\right] \text{ for } n = 1, 2, \ldots.$$

An easy inductive argument shows that each polynomial $P_n(x)$ has a constant term equal to zero. This guarantees that if $f \in \overline{\mathcal{A}}$, then $P_n(f) \in \overline{\mathcal{A}}$ for each n. Also, by Lemma 11.4, we know that the sequence $\{P_n(x)\}$ converges uniformly to $\sqrt{x}$ on $[0, 1]$. Thus, if $f \in \overline{\mathcal{A}}$ is non-zero, then put $c = \|f\|_\infty$, and note that:

1. The sequence $\left\{P_n\left(\frac{f^2}{c^2}\right)\right\} \subseteq \overline{\mathcal{A}}$ converges uniformly to $\frac{|f|}{c}$. Hence, $|f| \in \overline{\mathcal{A}}$.
2. Since $\left\{P_n\left(\frac{|f|}{c}\right)\right\} \subseteq \overline{\mathcal{A}}$ converges uniformly to $\sqrt{\frac{|f|}{c}}$, we see that $\sqrt{\frac{|f|}{c}} \in \overline{\mathcal{A}}$.

Thus, if $f \in \overline{\mathcal{A}}$, then both $|f|$ and $\sqrt{|f|}$ belong to $\overline{\mathcal{A}}$. In particular, $\overline{\mathcal{A}}$ is an algebra and a function space.

Now, suppose that $\overline{\mathcal{A}}$ is not of the form $\{f \in C(X)\colon\ f(a) = 0\}$ for some $a \in X$. This implies that for each $x \in X$, there exists some $f \in \overline{\mathcal{A}}$ with $f(x) \neq 0$. Thus, for each $x \in X$, there exists some $f_x \in \overline{\mathcal{A}}$ and a neighborhood V_x of x with $f_x(y) \neq 0$ for all $y \in V_x$. By the compactness of X, there exist $x_1, \ldots, x_n \in X$ with $X = \bigcup_{i=1}^n V_{x_i}$. Note that the function $g = f_{x_1}^2 + \cdots + f_{x_n}^2$ of $\overline{\mathcal{A}}$ satisfies $g(x) > 0$ for each $x \in X$. Multiplying by an appropriate constant, we can assume that $g(x) > 1$ holds for all x. Put $h_n = \sqrt[2^n]{g}$, and note that $h_n \in \overline{\mathcal{A}}$ and that $h_n(x) \downarrow 1$ for each $x \in X$. By Dini's theorem, $\{h_n\}$ converges uniformly to the constant function $\mathbf{1}$, and so $\mathbf{1} \in \overline{\mathcal{A}}$. Theorem 11.5 now guarantees that $\overline{\mathcal{A}} = C(X)$ must hold.

Problem 11.11. *Let $\mathcal{A}$ be the vector space generated by the functions*

$$\mathbf{1}, \sin x, \sin^2 x, \sin^3 x, \ldots$$

defined on $[0, 1]$. *That is, $f \in \mathcal{A}$ if and only if there is a non-negative integer k and real numbers $\alpha_0, \alpha_1, \ldots, \alpha_k$ (all depending on f) such that $f(x) = \sum_{n=0}^k \alpha_n \sin^n x$*

for each $x \in [0, 1]$. Show that A is an algebra and that A is dense in $C[0, 1]$ with respect to the uniform metric.

Solution. Clearly, A is an algebra of functions that contains the constant function 1. Also, since the function $f(x) = \sin x$ separates the points of $[0, 1]$, the algebra A likewise separates the points of $[0, 1]$. By the Stone–Weierstrass theorem, A is dense in $C[0, 1]$.

Problem 11.12. *Let X be a compact subset of $\mathbb{R}$. Show that $C(X)$ is a separable metric space (with respect to the uniform metric).*

Solution. The polynomials with rational coefficients form a countable set (why?). By Corollary 11.6, this set is dense in $C(X)$.

Problem 11.13. *Generalize the previous exercise as follows: Show that if (X, d) is a compact metric space, then $C(X)$ is a separable metric space.*

Solution. By Problem 7.2, we know that X is a separable metric space. Fix a countable dense subset $\{x_1, x_2, \ldots\}$ of X and for each n let $f_n: X \longrightarrow \mathbb{R}$ be the function defined by $f_n(t) = d(t, x_n)$ for each $t \in X$.

Now, let $x, y \in X$ satisfy $x \neq y$. Put $d(x, y) = 2\delta > 0$. Choose some n with $d(x, x_n) < \delta$, and note that

$$f_n(y) = d(y, x_n) \geq d(x, y) - d(x, x_n) \geq 2\delta - \delta = \delta > d(x, x_n) = f_n(x),$$

so that $f_n(x) \neq f_n(y)$. This implies that the algebra generated by $\{\mathbf{1}, f_1, f_2, \ldots\}$ separates the points of X. By the Stone–Weierstrass theorem (Theorem 11.5), this algebra must be dense in $C(X)$.

Next, consider the collection C of all finite products of the countable collection $\{\mathbf{1}, f_1, f_2, \ldots\}$ and note that C is a countable set, say $C = \{g_1, g_2, \ldots\}$. To complete the proof note that the finite linear combinations of $\{\mathbf{1}, g_1, g_2, \ldots\}$ with rational coefficients form a countable dense subset of $C(X)$.

Problem 11.14. *Let X and Y be two compact metric spaces. Consider the Cartesian product $X \times Y$ equipped with the distance D_1 given in Problem 7.4, so that $X \times Y$ is a compact metric space. Show that if $f \in C(X \times Y)$ and $\epsilon > 0$, then there exist functions $\{f_1, \ldots, f_n\} \subseteq C(X)$ and $\{g_1, \ldots, g_n\} \subseteq C(Y)$ such that*

$$\Big| f(x, y) - \sum_{i=1}^{n} f_i(x) g_i(y) \Big| < \epsilon$$

holds for all $(x, y) \in X \times Y$.

Solution. Consider the set

$$\mathcal{A} = \Big\{ h \in C(X \times Y)\colon\ \exists \{f_1, \ldots, f_n\} \subseteq C(X),\ \{g_1, \ldots, g_n\} \subseteq C(Y)$$
$$\text{with } h(x, y) = \sum_{i=1}^{n} f_i(x) g_i(y)\ \forall\, (x, y) \in X \times Y \Big\}.$$

Then, $\mathcal{A}$ is an algebra of functions of $C(X \times Y)$ and $\mathbf{1} \in \mathcal{A}$. On the other hand, if $(x_1, y_1) \neq (x_2, y_2)$, then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$, then select some $f \in C(X)$ with $f(x_1) \neq f(x_2)$, and let $F(x, y) = f(x)$ for all $(x, y) \in X \times Y$. If $y_1 \neq y_2$, then pick some $g \in C(Y)$ with $g(y_1) \neq g(y_2)$, and put $F(x, y) = g(y)$. In either case, $F \in \mathcal{A}$ and $F(x_1, y_1) \neq F(x_2, y_2)$ holds, so that $\mathcal{A}$ separates the points of $X \times Y$. Now, by the Stone–Weierstrass theorem (Theorem 11.5), we have $\overline{\mathcal{A}} = C(X \times Y)$, and the desired conclusion follows.

CHAPTER 3

THE THEORY OF MEASURE

12. SEMIRINGS AND ALGEBRAS OF SETS

Problem 12.1. *If X is a topological space, then show that the collection*

$$\mathcal{S} = \{C \cap O\colon C \text{ closed and } O \text{ open}\} = \{C_1 \setminus C_2\colon C_1,\ C_2 \text{ closed sets}\}$$

is a semiring of subsets of X.

Solution. From $\emptyset = \emptyset \cap \emptyset$ and $X = X \cap X$, we see that $\emptyset,\ X \in \mathcal{S}$. Next, notice that $C_1 \cap O_1,\ C_2 \cap O_2 \in \mathcal{S}$ imply

$$(C_1 \cap O_1) \cap (C_2 \cap O_2) = (C_1 \cap C_2) \cap (O_1 \cap O_2) \in \mathcal{S}.$$

Now, if $C_1 \cap O_1,\ C_2 \cap O_2 \in \mathcal{S}$, then

$$\begin{aligned} C_1 \cap O_1 \setminus C_2 \cap O_2 &= (C_1 \cap O_2) \cap (C_2 \cap O_2)^c \\ &= (C_1 \cap O_1) \cap (C_2^c \cup O_2^c) \\ &= (C_1 \cap O_1) \cap \left[C_2^c \cup (O_2^c \cap C_2)\right] \\ &= \left[C_1 \cap (O_1 \cap C_2^c)\right] \cup \left[(C_1 \cap C_2 \cap O_2^c) \cap O_1\right] \\ &= A \cup B, \end{aligned}$$

where $A = C_1 \cap (O_1 \cap C_2^c) \in \mathcal{S}$ and $B = (C_1 \cap C_2 \cap O_2^c) \cap O_1 \in \mathcal{S}$ satisfy $A \cap B = \emptyset$.

Problem 12.2. *Let $\mathcal{S}$ be a semiring of subsets of a set X, and let $Y \subseteq X$. Show that $\mathcal{S}_Y = \{Y \cap A\colon A \in \mathcal{S}\}$ is a semiring of Y (called the* **restriction semiring** *of $\mathcal{S}$ to Y).*

Solution. The conclusion follows from the identities:

a. $Y \cap \emptyset = \emptyset$;
b. $(Y \cap A) \cap (Y \cap B) = Y \cap (A \cap B)$; and
c. $Y \cap A \setminus Y \cap B = Y \cap (A \setminus B)$.

Problem 12.3. *Let $\mathcal{S}$ be the collection of all subsets of $[0, 1)$ that can be written as finite unions of subsets of $[0, 1)$ of the form $[a, b)$. Show that $\mathcal{S}$ is an algebra of sets but not a σ-algebra.*

Solution. Let $A = \bigcup_{i=1}^{n}[a_i, b_i)$ and $B = \bigcup_{j=1}^{m}[c_j, d_j)$. Then, we have

a. $A \cup B \in \mathcal{S}$;
b. $A \cap B = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}[a_i, b_i) \cap [c_j, d_j) \in \mathcal{S}$; and
c. $[0, 1) \setminus A = \bigcap_{i=1}^{n}\big([0, 1) \setminus [a_i, b_i)\big) \in \mathcal{S}$, where the last membership holds since each $[0, 1) \setminus [a_i, b_i)$ can be written as a finite union of sets of the form $[a, b)$.

To see that $\mathcal{S}$ is not a σ-algebra note that $\bigcap_{n=1}^{\infty}[0, \frac{1}{n}) = \{0\} \notin \mathcal{S}$.

Problem 12.4. *Prove that the σ-sets of the semiring*

$$\mathcal{S} = \{\, [a, b)\colon\ a, b \in \mathbf{R}\,\}$$

form a topology for the real numbers.

Solution. Let τ be the collection of all σ-sets of $\mathcal{S}$. Clearly, $\emptyset \in \tau$ and $\mathbf{R} = \bigcup_{n=1}^{\infty}[-n, n) \in \tau$. It should be clear that τ is closed under finite intersections. Thus, in order to establish that τ is a topology, we need to show that τ is closed under arbitrary unions. That is, if $\{[a_i, b_i)\colon\ i \in I\}$ is a collection of nonempty members of $\mathcal{S}$, then we must show that $A = \bigcup_{i \in I}[a_i, b_i)$ belongs to τ (i.e., that A is a σ-set).

To see this, let $V = \bigcup_{i \in I}(a_i, b_i)$. Then, V is an open set, and thus, there exists an at-most countable collection of pairwise disjoint open interval $\{(c_j, d_j)\colon\ j \in J\}$ (see part (g) of Problem 6.11) such that $V = \bigcup_{j \in J}(c_j, d_j)$. For each $j \in J$, let $A_j = [c_j, d_j)$ if $c_j = a_i$ for some $i \in I$ and let $A_j = (c_j, d_j)$ if $c_j \neq a_i$ for all $i \in I$. Clearly, each A_j is a σ-set. Moreover, it is easy to see that $A = \bigcup_{j \in J} A_j$ holds, which shows that A is a σ-set.

Problem 12.5. *Let $\mathcal{S}$ be a semiring of subsets of a nonempty set X. What additional requirements must be satisfied for $\mathcal{S}$ to be a base for a topology on X? (For the definition of a base see Problem 8.18.) Prove that if such is the case, then each member of $\mathcal{S}$ is both open and closed in this topology.*

Solution. Since $\mathcal{S}$ is already closed under finite intersections, it follows from the definition of a base that $\mathcal{S}$ will be a base if and only if $\bigcup_{A\in\mathcal{S}} A = X$.

Now, assume that $\bigcup_{A\in\mathcal{S}} A = X$ holds. Note first that if $A,\ B \in \mathcal{S}$, then (since $\mathcal{S}$ is a semiring) $A \setminus B$ can be written as a finite union of (disjoint) members of $\mathcal{S}$. It follows that $A \setminus B$ belongs to the topology generated by $\mathcal{S}$. Thus, if $B \in \mathcal{S}$, then the relation

$$B^c = X \setminus B = \Big(\bigcup_{A\in\mathcal{S}} A\Big) \setminus B = \bigcup_{A\in\mathcal{S}} (A \setminus B),$$

shows that B^c belongs to the topology generated by $\mathcal{S}$. That is, in this case, every $B \in \mathcal{S}$ is a closed and open set.

Problem 12.6. *Let A be a fixed subset of a set X. Determine the two σ-algebras of subsets of X generated by*

a. $\{A\}$, *and*
b. $\{B\colon\ A \subseteq B \subseteq X\}$.

Solution. (a) $\{\emptyset,\ A,\ A^c,\ X\}$ and (b) $\{B\colon\ A \subseteq B \text{ or } A \subseteq B^c\}$.

Problem 12.7. *Let X be an uncountable set, and let*

$$\mathcal{S} = \{E \subseteq X\colon\ E \textit{ or } E^c \textit{ is at-most countable}\}.$$

Show that $\mathcal{S}$ is the σ-algebra generated by the one-point subsets of X.

Solution. Clearly, $\mathcal{S}$ contains the one-point subsets of X, and every member of $\mathcal{S}$ must be a member of the σ-algebra generated by the one-point sets. Thus, it remains to be shown that $\mathcal{S}$ is a σ-algebra.

Clearly, $\emptyset,\ X \in \mathcal{S}$ and $\mathcal{S}$ is closed under complementation. Then let $\{A_n\} \subseteq \mathcal{S}$. If each A_n is at-most countable, then $\bigcup_{n=1}^{\infty} A_n$ is at-most countable, and consequently $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$. On the other hand, if some A_k is uncountable, then $(A_k)^c$ is at-most countable and the inclusion $\big(\bigcup_{n=1}^{\infty} A_n\big)^c \subseteq (A_k)^c$ shows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.

Problem 12.8. *Characterize the metric spaces whose open sets form a σ-algebra.*

Solution. We shall show that the open sets of a metric space X form a σ-algebra if and only if X is a discrete metric space (i.e., if and only if every subset of X is open).

Let τ be the collection of all open sets. If $\tau = \mathcal{P}(X)$, then clearly τ is a σ-algebra. On the other hand, if τ is a σ-algebra and $x \in X$, then

$\{x\} = \bigcap_{n=1}^{\infty} B(x, \frac{1}{n})$ shows that $\{x\}$ is an open set. This easily implies that every subset of X is open (i.e., $\tau = \mathcal{P}(X)$ holds).

Problem 12.9. *Determine the σ-algebra generated by the nowhere dense subsets of a topological space.*

Solution. Let X be a topological space. Define

$$\mathcal{A} = \{A \subseteq X\colon\ A \text{ is meager or } A^c \text{ is meager}\}.$$

Recall that a set is called meager if it can be written as a countable union of nowhere dense sets—a set A is nowhere dense if $(\overline{A})^o = \emptyset$. We claim that $\mathcal{A}$ is the σ-algebra generated by the nowhere dense sets of X. Clearly, every nowhere dense set belongs to $\mathcal{A}$, and every member of $\mathcal{A}$ belongs to the σ-algebra generated by the nowhere dense sets. So, it suffices to establish that $\mathcal{A}$ is a σ-algebra of sets.

Clearly, $\emptyset$, $X \in \mathcal{A}$. Also, it should be obvious that $\mathcal{A}$ is closed under complementation. Now, let $\{A_n\} \subseteq \mathcal{A}$. If each A_n is meager, then clearly $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. On the other hand, if $(A_k)^c$ is a meager set for some k, then the set inclusion $\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq (A_k)^c$ implies $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Therefore, $\mathcal{A}$ is a σ-algebra.

Problem 12.10. *Let X be a nonempty set, and let $\mathcal{F}$ be an uncountable collection of subsets of X. Show that any element of the σ-algebra generated by $\mathcal{F}$ belongs to the σ-algebra generated by some countable subcollection of $\mathcal{F}$.*

Solution. Assume $\mathcal{F}$ to be uncountable. Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{F}$. Denote by $\{\mathcal{A}_i\colon i \in I\}$ the family of all σ-algebras each of which is generated by a countable subset of $\mathcal{F}$. It suffices to show that $\mathcal{B} = \bigcup_{i\in I} \mathcal{A}_i$ is a σ-algebra (because if this is the case, then $\mathcal{A} = \mathcal{B}$ must hold, and the conclusion follows).

Clearly, $\emptyset \in \mathcal{B}$. Also, if $A \in \mathcal{B}$, then it is easy to see that $A^c \in \mathcal{B}$ likewise holds. Now, let $\{A_n\} \subseteq \mathcal{B}$. Since each A_n belongs to a σ-algebra generated by a countable subset of $\mathcal{F}$, it easily follows that there exists some $i \in I$ with $\{A_n\} \subseteq \mathcal{A}_i$. Thus, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}_i \subseteq \mathcal{B}$. That is, $\mathcal{B}$ is a σ-algebra, as required.

Problem 12.11. *Show that every F_σ- and every G_δ-subset of a topological space is a Borel set.*

Solution. The Borel sets are the members of the σ-algebra generated by the open sets. So, a countable intersection of open sets (or a countable union of closed sets) is always a Borel set.

Problem 12.12. *Show that every infinite σ-algebra of sets has uncountably many sets.*

Solution. Let $\mathcal{A}$ be an infinite σ-algebra of subsets of a set X. If $\mathcal{A}$ contains a sequence $\{A_n\}$ of nonempty pairwise disjoint sets, then $\mathcal{A}$ has uncountably many members. Indeed, if this is the case, then for each subset s of natural numbers let $A_s = \bigcup_{n \in s} A_n \in \mathcal{A}$, and note that $A_s \neq A_t$ if $s \neq t$. By Problem 5.6, the collection $\{A_s\colon s \in \mathcal{P}(\mathbb{N})\}$ has uncountably many members, and so $\mathcal{A}$ must likewise have uncountably many members.

Next, we shall show that there exists a sequence $\{B_n\} \subseteq \mathcal{A}$ with $B_{n+1} \subseteq B_n$ and $B_{n+1} \neq B_n$ for all n. If this is done, then put $A_n = B_n \setminus B_{n+1}$, and use the above arguments to see that $\mathcal{A}$ is an uncountable set.

Using induction, we shall establish the existence of a sequence $\{B_n\}$ such that:

1. $B_{n+1} \subseteq B_n$ and $B_{n+1} \neq B_n$ for all n, and
2. $\{B_n \cap A\colon\ A \in \mathcal{A}\}$ is an infinite set.

The basic step of the induction is the following: Assume that $B_n \in \mathcal{A}$ has been chosen so that $\{B_n \cap A\colon\ A \in \mathcal{A}\}$ has infinitely many members. Choose $C \in \mathcal{A}$ so that $\emptyset \subseteq B_n \cap C \subseteq B_n$ is a proper inclusion at both ends. In view of

$$B_n \cap A = [(B_n \cap C) \cap A] \cup [(B_n \setminus C) \cap A],$$

we see that either $\{(B_n \cap C) \cap A\colon\ A \in \mathcal{A}\}$ or $\{(B_n \setminus C) \cap A\colon\ A \in \mathcal{A}\}$ is infinite. If $\{(B_n \cap C) \cap A\colon\ A \in \mathcal{A}\}$ is infinite, put $B_{n+1} = B_n \cap C$. If $\{(B_n \cap C) \cap A\colon\ A \in \mathcal{A}\}$ is finite, put $B_{n+1} = B_n \setminus C$.

Start the induction with $B_1 = X$.

Problem 12.13. *Let (X, τ) be a topological space, let $\mathcal{B}$ be the σ-algebra of its Borel sets, and let Y be an arbitrary subset of X. If Y is considered equipped with the induced topology and $\mathcal{B}_Y$ denotes the σ-algebra of Borel sets of (Y, τ), then show that*

$$\mathcal{B}_Y = \{A \cap Y\colon\ A \in \mathcal{B}\}.$$

Solution. Let (X, τ), Y, and $\mathcal{B}_Y$ be as in the problem, and let

$$\mathcal{A} = \{A \cap Y\colon\ A \in \mathcal{B}\}.$$

We have to show that $\mathcal{B}_Y \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{B}_Y$ both hold.

Clearly, $\mathcal{A}$ is a σ-algebra of subsets of Y and $O \cap Y \in \mathcal{A}$ holds for each $O \in \tau$. Thus, $\mathcal{A}$ contains the open sets of Y, and so $\mathcal{B}_Y \subseteq \mathcal{A}$. Now, consider the collection of sets

$$\mathcal{C} = \{A \in \mathcal{B}\colon\ A \cap Y \in \mathcal{B}_Y\}.$$

It is easy to see that C is a σ-algebra of subsets of X satisfying $\tau \subseteq C$. Hence, $C = \mathcal{B}$, and this implies that $\mathcal{A} \subseteq \mathcal{B}_Y$. Therefore, $\mathcal{B}_Y = \mathcal{A}$, as claimed.

Problem 12.14. *Let $A_1, \ldots, A_n$ be sets in some semiring $\mathcal{S}$. Show that there exists a finite number of pairwise disjoint sets $B_1, \ldots, B_m$ of $\mathcal{S}$ such that each A_i can be written as a union of sets from the $B_1, \ldots, B_m$.*

Solution. We use induction on n. For $n = 1$ the result is trivial. Thus, assume that the result is true for some n, and let $A_1, \ldots, A_n, A_{n+1}$ be members of $\mathcal{S}$. Pick a finite number of pairwise disjoint members $B_1, \ldots, B_m$ of $\mathcal{S}$ such that each A_i, $1 \le i \le n$, can be written as a union of sets from $B_1, \ldots, B_m$. Clearly, $\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{j=1}^{m} B_j$. The sets $B_1 \cap A_{n+1}, \ldots, B_m \cap A_{n+1}$ are pairwise disjoint members of $\mathcal{S}$. On the other hand, for each $1 \le i \le m$ there exists a finite pairwise disjoint collection $\mathcal{F}_i \subseteq \mathcal{S}$ with $B_i \setminus A_{n+1} = \bigcup_{C \in \mathcal{F}_i} C$ (by the definition of the semiring). Thus, the collection

$$\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_m \cup \{B_1 \cap A_{n+1}, \ldots, B_m \cap A_{n+1}\} \subseteq \mathcal{S}$$

is finite and pairwise disjoint. Moreover, each A_i $(1 \le i \le n)$ can be written as a union of members of $\mathcal{F}$. Now, observe that

$$A_{n+1} = \Big(A_{n+1} \setminus \bigcup_{j=1}^{m} B_j\Big) \cup (B_1 \cap A_{n+1}) \cup \cdots \cup (B_m \cap A_{n+1}).$$

By Theorem 12.2(1) there exist pairwise disjoint sets $D_1, \ldots, D_k$ in $\mathcal{S}$ such that $A_{n+1} \setminus \bigcup_{j=1}^{m} B_j = \bigcup_{r=1}^{k} D_r$. Finally, the collection $\mathcal{F} \cup \{D_1, \ldots, D_k\} \subseteq \mathcal{S}$ is finite and pairwise disjoint, and each set A_i $(1 \le i \le n+1)$ can be written as a union from these sets.

13. MEASURES ON SEMIRINGS

Problem 13.1. *Let $\{a_n\}$ be a sequence of non-negative real numbers. Let $\mu(\emptyset) = 0$, and for every nonempty subset A of $\mathbb{N}$ put $\mu(A) = \sum_{n \in A} a_n$. Show that $\mu \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a measure.*

Solution. If $\{A_n\}$ is a sequence of pairwise disjoint subsets of $\mathbb{N}$ and $A = \bigcup_{n=1}^{\infty} A_n$, then note that

$$\mu(A) = \sum_{k \in A} a_k = \sum_{n=1}^{\infty} \Big(\sum_{k \in A_n} a_k\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

Problem 13.2. *Let $\mathcal{S}$ be a semiring, and let $\mu: \mathcal{S} \to [0, \infty]$ be a set function such that $\mu(A) < \infty$ for some $A \in \mathcal{S}$. If μ is σ-additive, then show that μ is a measure.*

Solution. Write $A = A \cup \emptyset \cup \emptyset \cup \cdots$. Then,

$$\mu(A) = \mu(A) + \mu(\emptyset) + \mu(\emptyset) + \cdots .$$

If $\mu(\emptyset) > 0$, then $\mu(A) = \infty$, contrary to our hypothesis. Thus, $\mu(\emptyset) = 0$, and so μ is a measure.

Problem 13.3. *Let X be an uncountable set, and let the σ-algebra*

$$\mathcal{S} = \{E \subseteq X: E \text{ or } E^c \text{ is at-most countable}\};$$

see also Problem 12.7. Show that $\mu: \mathcal{S} \to [0, \infty)$, defined by $\mu(E) = 0$ if E is at-most countable and $\mu(E) = 1$ if E^c is at-most countable, is a measure on $\mathcal{S}$.

Solution. Clearly, $\mu(\emptyset) = 0$. For the σ-additivity of μ let $\{E_n\} \subseteq \mathcal{S}$ be a pairwise disjoint sequence. Let $E = \bigcup_{n=1}^{\infty} E_n$. If each E_n is at-most countable, then E itself is at-most countable, and so $\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) = 0$ holds. On the other hand, if E_k^c is at-most countable for some k, then (in view of $E_n \cap E_k = \emptyset$ for $n \neq k$) we must have $E_n \subseteq E_k^c$ for $n \neq k$, and so E_n is at-most countable for each $n \neq k$. Thus,

$$1 = \mu(E) = \mu(E_k) = \sum_{n=1}^{\infty} \mu(E_n).$$

It is interesting to observe that if $X = [0, 1]$, then $\mathcal{S}$ is a σ-subalgebra of the Lebesgue measurable subsets of $[0, 1]$, and μ is the restriction of the Lebesgue measure to $\mathcal{S}$.

Problem 13.4. *Let X be a nonempty set, and let $f: X \to [0, \infty]$ be a function. Define $\mu: \mathcal{P}(X) \to [0, \infty]$ by $\mu(A) = \sum_{x \in A} f(x)$ if $A \neq \emptyset$ and is at-most countable, $\mu(A) = \infty$ if A is uncountable, and $\mu(\emptyset) = 0$. Show that μ is a measure.*

Solution. For the σ-additivity of μ, let $\{A_n\}$ be a pairwise disjoint sequence of subsets of X. Let $A = \bigcup_{n=1}^{\infty} A_n$. If some A_n is uncountable, then A is likewise uncountable, and hence, in this case $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n) = \infty$ holds. On the

other hand, if each A_n is at-most countable, then A is also at-most countable, and so

$$\mu(A) = \sum_{x \in A} f(x) = \sum_{n=1}^{\infty}\Big[\sum_{x \in A_n} f(x)\Big] = \sum_{n=1}^{\infty} \mu(A_n)$$

also holds.

Problem 13.5. *Let $\mathcal{S}$ be a semiring, and let $\mu \colon \mathcal{S} \to [0, \infty]$ be a finitely additive measure. Show that if μ is σ-subadditive, then μ is a measure.*

Solution. Let $\{A_n\} \subseteq \mathcal{S}$ be a pairwise disjoint such that $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$. By hypothesis, $\mu(A) \le \sum_{n=1}^{\infty} \mu(A_n)$ holds. On the other hand, if k is fixed, then there exist pairwise disjoint sets $B_1, \dots, B_m \in \mathcal{S}$ such that $A \setminus \bigcup_{n=1}^{k} A_n = \bigcup_{i=1}^{m} B_i$ (see Theorem 12.2). Since $A = \big[\bigcup_{n=1}^{k} A_n\big] \cup \big[\bigcup_{i=1}^{m} B_i\big]$ is a finite union of pairwise disjoint members of $\mathcal{S}$, the finite additivity of μ implies

$$\sum_{n=1}^{k} \mu(A_n) \le \sum_{n=1}^{k} \mu(A_n) + \sum_{i=1}^{m} \mu(B_i) = \mu(A).$$

Since k is arbitrary, $\sum_{n=1}^{\infty} \mu(A_n) \le \mu(A)$ also holds, and so μ is a measure.

Problem 13.6. *Let $\{\mu_n\}$ be an increasing sequence of measures on a semiring $\mathcal{S}$; that is, $\mu_n(A) \le \mu_{n+1}(A)$ holds for all $A \in \mathcal{S}$ and all n. Define $\mu \colon \mathcal{S} \to [0, \infty]$ by $\mu(A) = \sup\{\mu_n(A)\}$ for each $A \in \mathcal{S}$. Show that μ is a measure.*

Solution. Clearly, $\mu(\emptyset) = 0$. Now, let $\{A_n\} \subseteq \mathcal{S}$ be a pairwise disjoint sequence such that $\bigcup_{n=1}^{\infty} A_n = A \in \mathcal{S}$. Since each μ_i is a measure,

$$\mu_i(A) = \sum_{n=1}^{\infty} \mu_i(A_n) \le \sum_{n=1}^{\infty} \mu(A_n)$$

holds, and so $\mu(A) \le \sum_{n=1}^{\infty} \mu(A_n)$. On the other hand, for each k we have

$$\sum_{n=1}^{k} \mu(A_n) = \lim_{i\to\infty} \sum_{n=1}^{k} \mu_i(A_n) = \lim_{i\to\infty} \mu_i\Big(\bigcup_{n=1}^{k} A_n\Big) \le \mu(A).$$

Thus, $\sum_{n=1}^{\infty} \mu(A_n) \le \mu(A)$ also holds, which shows that the measure μ is σ-additive.

Problem 13.7. *Consider the semiring $\mathcal{S} = \{A \subseteq \mathbb{R}\colon A$ is at-most countable$\}$, and define $\mu\colon \mathcal{S} \to [0, \infty]$ by $\mu(A) = 0$ if A is finite and $\mu(A) = \infty$ if A is countable. Show that μ is a finitely additive measure that is not a measure.*

Solution. Let $A_1, \dots, A_n$ be pairwise disjoint members of $\mathcal{S}$. Put $A = \bigcup_{i=1}^n A_i$. If each A_i is a finite set, then A is likewise a finite set, and $\sum_{i=1}^n \mu(A_i) = \mu(A) = 0$ holds. On the other hand, if one of the A_i is countable, then A itself is also countable, and $\sum_{i=1}^n \mu(A_i) = \mu(A) = \infty$ holds. Thus, μ is a finitely additive measure.

To see that μ is not σ-additive, note that $\mathbb{N} = \bigcup_{n=1}^\infty \{n\}$, while

$$0 = \sum_{n=1}^{\infty} \mu(\{n\}) < \mu(\mathbb{N}) = \infty.$$

Problem 13.8. *Show that every finitely additive measure is monotone.*

Solution. Assume that $\mu\colon \mathcal{S} \longrightarrow [0, \infty]$ is a finitely additive measure. Let $A, B \in \mathcal{S}$ satisfy $A \subseteq B$. Choose a finite collection of disjoint sets $C_1, \dots, C_n$ of $\mathcal{S}$ such that $B \setminus A = \bigcup_{i=1}^n C_i$. Then,

$$B = A \cup C_1 \cup \cdots \cup C_n$$

is a finite union of pairwise disjoint sets of $\mathcal{S}$. Thus, by the finite additivity of μ, we have

$$\mu(A) \le \mu(A) + \mu(C_1) + \cdots + \mu(C_n) = \mu(B).$$

Problem 13.9. *Consider the set function μ defined in Example 13.6. That is, consider a nondecreasing and left-continuous function $f\colon \mathbb{R} \to \mathbb{R}$ and then define the set function $\mu\colon \mathcal{S} \to [0, \infty)$ by $\mu([a, b)) = f(b) - f(a)$, where $\mathcal{S}$ is the semiring $\mathcal{S} = \{[a, b)\colon -\infty < a \le b < \infty\}$. Prove alternately the fact that μ is a measure.*

Solution. An alternate way of proving the σ-additivity of μ is as follows. Let $a < b$ and let $[a, b) = \bigcup_{n=1}^\infty [a_n, b_n)$ with the sequence $\{[a_n, b_n)\}$ pairwise disjoint. For each $a < x \le b$ let

$$s_x = \sum_i [f(b_i) - f(a_i)],$$

where the sum (possibly a series) extends over all i for which $[a_i, b_i) \subseteq [a, x)$ holds; we let $s_x = 0$ if there is no such interval. Since f is nondecreasing, we

have $s_x \leq f(x) - f(a)$. Next, note that the set

$$A = \{x \in (a, b]:\ s_x = f(x) - f(a)\}$$

is nonempty. Let $t = \sup A$, and note that $a < t \leq b$. Now, for $x \in A$, we have

$$f(x) - f(a) = s_x \leq s_t \leq f(t) - f(a),$$

and so, by the left-continuity of f, we get $s_t = f(t) - f(a)$. That is, $t \in A$.

Our objective is to establish that $t = b$ holds. Assume by way of contradiction that $a < t < b$. Then $a_k \leq t < b_k$ must hold for some k. Since the sequence $\{[a_n, b_n)\}$ is pairwise disjoint, observe that $[a_i, b_i) \subseteq [a, t)$ holds if and only if $[a_i, b_i) \subseteq [a, a_k)$. Thus, $s_t = s_{a_k}$ holds. In particular, the relation

$$f(t) - f(a) = s_t = s_{a_k} \leq f(a_k) - f(a) \leq f(t) - f(a)$$

guarantees that $a_k \in A$. However, this implies $b_k \in A$, which is impossible. Therefore, $t = b$ holds, which guarantees that

$$\mu([a, b)) = \sum_{n=1}^{\infty} \mu([a_n, b_n)).$$

14. OUTER MEASURES AND MEASURABLE SETS

Problem 14.1. *Show that a countable union of null sets is again a null set.*

Solution. The conclusion follows from the inequality

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Problem 14.2. *If μ is an outer measure on a set X and A is a null set, then show that*

$$\mu(B) = \mu(A \cup B) = \mu(B \setminus A)$$

holds for every subset B of X.

Solution. The conclusion follows from the inequalities:

$$\begin{aligned}\mu(B) &\le \mu(B \cup A) = \mu\big((B \setminus A) \cup A\big)\\ &\le \mu(B \setminus A) + \mu(A) = \mu(B \setminus A) \le \mu(B).\end{aligned}$$

Problem 14.3. *Let μ be an outer measure on a set X. If a sequence $\{A_n\}$ of subsets of X satisfies $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, then show that the set*

$$E = \big\{x \in X\colon\ x \text{ belongs to } A_n \text{ for infinitely many } n\big\}$$

is a null set.

Solution. Assume that a sequence $\{A_n\}$ satisfies $\sum_{n=1}^{\infty}\mu(A_n) < \infty$. For each n let $E_n = \bigcup_{i=n}^{\infty} A_i$, and note that $E \subseteq E_n$ holds for each n. Therefore,

$$0 \le \mu(E) \le \mu(E_n) \le \sum_{i=n}^{\infty} \mu(A_i) \longrightarrow 0,$$

and hence $\mu(E) = 0$.

Problem 14.4. *If E is a measurable subset of X, then show that for every subset A of X the following equality holds:*

$$\mu(E \cap A) + \mu(E \cup A) = \mu(E) + \mu(A).$$

Solution. The measurability of E gives

$$\mu(E \cup A) = \mu\big((E \cup A) \cap E\big) + \mu\big((E \cup A) \cap E^c\big) = \mu(E) + \mu(A \cap E^c).$$

Consequently, we have

$$\mu(E \cup A) + \mu(E \cap A) = \mu(E) + \mu(A \cap E^c) + \mu(A \cap E) = \mu(E) + \mu(A).$$

Problem 14.5. *Let μ be an outer measure on a set X. If A is a nonmeasurable subset of X and E is a measurable set such that $A \subseteq E$, then show that $\mu(E \setminus A) > 0$.*

Solution. If $\mu(E \setminus A) = 0$ holds, then $E \setminus A \in \Lambda$. Thus, $A = E \setminus (E \setminus A) \in \Lambda$, which is a contradiction. Therefore, $\mu(E \setminus A) > 0$.

Problem 14.6. *Let A be a subset of X, and let $\{E_n\}$ be a disjoint sequence of measurable sets. Show that*

$$\mu\Big(\bigcup_{n=1}^{\infty} A \cap E_n\Big) = \sum_{n=1}^{\infty} \mu(A \cap E_n).$$

Solution. From the σ-subadditivity of μ, we see that

$$\mu\Big(A \cap \Big[\bigcup_{n=1}^{\infty} E_n\Big]\Big) = \mu\Big(\bigcup_{n=1}^{\infty} A \cap E_n\Big) \le \sum_{n=1}^{\infty} \mu(A \cap E_n).$$

On the other hand, Lemma 14.5 implies

$$\sum_{n=1}^{k} \mu(A \cap E_n) = \mu\Big(A \cap \Big[\bigcup_{n=1}^{k} E_n\Big]\Big) \le \mu\Big(A \cap \Big[\bigcup_{n=1}^{\infty} E_n\Big]\Big)$$

for each k, and so $\sum_{n=1}^{\infty} \mu(A \cap E_n) \le \mu\big(A \cap \big[\bigcup_{n=1}^{\infty} E_n\big]\big)$ also holds.

Problem 14.7. *Let $\{A_n\}$ be a sequence of subsets of X. Assume that there exists a disjoint sequence $\{B_n\}$ of measurable sets such that $A_n \subseteq B_n$ holds for each n. Show that*

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

Solution. Put $A = \bigcup_{n=1}^{\infty} A_n$ and note that $A \cap B_n = A_n$ holds for each n. Thus, using the preceding problem, we see that

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \mu\Big(A \cap \Big[\bigcup_{n=1}^{\infty} B_n\Big]\Big) = \sum_{n=1}^{\infty} \mu(A \cap B_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Problem 14.8. *Let μ be an outer measure on a set X. Show that a subset E of X is measurable if and only if for each $\epsilon > 0$ there exists a measurable set F such that $F \subseteq E$, and $\mu(E \setminus F) < \epsilon$.*

Solution. If E is measurable, then $F = E$ satisfies the condition for each $\varepsilon > 0$. For the converse, assume that the condition is satisfied.

Start by choosing for each n a measurable set F_n with $F_n \subseteq E$ and $\mu(E \setminus F_n) < \frac{1}{n}$. Put $F = \bigcup_{n=1}^{\infty} F_n \subseteq E$, and note that F is measurable. Consequently, $\mu(E \setminus F) \le \mu(E \setminus F_n) < \frac{1}{n}$ for each n implies $\mu(E \setminus F) = 0$, and so $E \setminus F$ is measurable. The measurability of E now follows from the identity $E = F \cup (E \setminus F)$.

An alternate proof of the preceding part goes as follows. Let A be a subset of X with $\mu(A) < \infty$. If $\epsilon > 0$ is given, pick a measurable subset F with $F \subseteq E$ and $\mu(E \setminus F) < \epsilon$. Then

$$\begin{aligned}\mu(A \cap E) &= \mu\big(A \cap [F \cup (E \setminus F)]\big) \\ &\le \mu(A \cap F) + \mu\big(A \cap (E \setminus F)\big) \le \mu(A \cap F) + \epsilon\end{aligned}$$

implies $\mu(A \cap F) - \mu(A \cap E) > -\epsilon$, and so

$$\begin{aligned}\mu(A) &= \mu(A \cap F) + \mu(A \cap F^c) \\ &\ge \mu(A \cap F) + \mu(A \cap E^c) \\ &= \mu(A \cap E) + +\mu(A \cap E^c) + \big[\mu(A \cap F) - \mu(A \cap E)\big] \\ &> \mu(A \cap E) + \mu(A \cap E^c) - \epsilon\end{aligned}$$

for all $\epsilon > 0$. This implies $\mu(A) \ge \mu(A \cap E) + \mu(A \cap E^c)$, which shows that E is a measurable set.

Problem 14.9. *Let μ be an outer measure on a set X. Assume that a subset E of X has the property that for each $\epsilon > 0$, there exists a measurable set F such that $\mu(E \Delta F) < \epsilon$. Show that E is a measurable set.*

Solution. Let $\varepsilon > 0$. According to the preceding problem, it suffices to show that $\mu(E \setminus G) < \varepsilon$ holds for some measurable set G with $G \subseteq E$.

For each n choose $F_n \in \Lambda$ with $\mu(E \Delta F_n) < 2^{-n}\varepsilon$. Put $F = \bigcap_{n=1}^{\infty} F_n \in \Lambda$. Since $F \setminus E \subseteq F_n \setminus E$ holds, we have

$$\mu(F \setminus E) \le \mu(F_n \setminus E) < 2^{-n}\varepsilon$$

for each n, and so $\mu(F \setminus E) = 0$. Thus, $F \setminus E \in \Lambda$, and hence $F \cap E = F \setminus (F \setminus E)$ is also a measurable set. Now, note that $F \cap E \subseteq E$ holds and

$$\mu(E \setminus E \cap F) = \mu(E \setminus F) = \mu\Big(\bigcup_{n=1}^{\infty}(E \setminus F_n)\Big) \le \sum_{n=1}^{\infty} \mu(E \setminus F_n) < \varepsilon.$$

Problem 14.10. *Let $X = \{1, 2, 3\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}\}$ and consider the set function $\mu\colon \mathcal{F} \to [0, \infty]$ defined by $\mu(\emptyset) = 0$, $\mu(\{1\}) = 2$ and $\mu(\{1, 2\}) = 1$.*

a. *Describe the outer measure μ^* generated by the set function μ.*
b. *Describe the σ-algebra of all μ^*-measurable subsets of X (and conclude that the set $\{1\} \in \mathcal{F}$ is not a measurable set).*

Solution. (a) The outer measure $\mu^*: \mathcal{P}(X) \to [0, \infty]$ is given by

$$\mu^*(\emptyset) = 0, \quad \mu^*(\{1\}) = \mu^*(\{2\}) = 1, \quad \mu^*(\{3\}) = \infty,$$
$$\mu^*(\{1, 2\}) = 1, \quad \mu^*(\{1, 3\}) = \mu^*(\{2, 3\}) = \mu^*(\{1, 2, 3\}) = \infty.$$

(b) The σ-algebra of all measurable sets is $\Lambda = \{\emptyset, \{3\}, \{1, 2\}, X\}$.

Problem 14.11. *Let $\nu: \mathcal{P}(X) \to [0, \infty]$ be a set function. Show that ν is an outer measure if and only if there exist a collection $\mathcal{F}$ of subsets of X containing the empty set and a set function $\mu: \mathcal{F} \to [0, \infty]$ with $\mu(\emptyset) = 0$ satisfying $\nu(A) = \mu^*(A)$ for all $A \in \mathcal{P}(X)$.*

Solution. Assume first that $\nu: \mathcal{P}(X) \to [0, \infty]$ is an outer measure. Let $\mathcal{F} = \mathcal{P}(X)$ and $\mu = \nu$. We claim that $\nu(A) = \mu^*(A)$ holds for each $A \in \mathcal{P}(X)$, where

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n)\colon \{A_n\} \subseteq \mathcal{F} \text{ and } A \subseteq \bigcup_{n=1}^{\infty} A_n\right\},$$

and $\inf \emptyset = \infty$. To see this, let $A \in \mathcal{P}(X)$. From $A = A \cup \emptyset \cup \emptyset \cup \emptyset \cdots$, we see that $\mu^*(A) \le \mu(A) = \nu(A)$. On the other hand, if $A \subseteq \bigcup_{n=1}^{\infty} A_n$ holds true, then from the σ-subadditivity of ν, we see that

$$\nu(A) \le \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu(A_n),$$

and so $\nu(A) \le \mu^*(A)$ is also true. Hence, $\nu(A) = \mu^*(A)$ for each subset A of X.

For the converse, assume that the outer measure μ^* generated by a set function $\mu: \mathcal{F} \to [0, \infty]$ satisfies $\mu(\emptyset) = 0$ and $\nu(A) = \mu^*(A)$ for each $A \in \mathcal{P}(X)$. We shall show that ν is an outer measure by verifying the three properties required to be satisfied by ν in order to be a measure.

(1) From $0 \le \nu(\emptyset) = \mu^*(\emptyset) \le \mu(\emptyset) + \mu(\emptyset) + \mu(\emptyset) + \cdots = 0$, we see that $\nu(\emptyset) = 0$.

(2) (*Monotonicity*) Let $A \subseteq B$ and let $\{A_n\}$ be a sequence of $\mathcal{F}$ with $B \subseteq \bigcup_{n=1}^{\infty} A_n$. Then, $A \subseteq \bigcup_{n=1}^{\infty} A_n$, and so $\mu^*(A) \le \sum_{n=1}^{\infty} \mu(A_n)$. Therefore,

$$\nu(A) = \mu^*(A) \le \inf\left\{\sum_{n=1}^{\infty} \mu(A_n)\colon \{A_n\} \subseteq \mathcal{F} \text{ and } B \subseteq \bigcup_{n=1}^{\infty} A_n\right\} = \mu^*(B) = \nu(B).$$

(If there is no sequence $\{A_n\}$ of $\mathcal{F}$ with $B \subseteq \bigcup_{n=1}^{\infty} A_n$, then $\mu^*(B) = \infty$, and $\nu(A) \le \nu(B) = \mu^*(B)$ is trivially true.)

(3) (σ-*Subadditivity*) Let $\{E_n\}$ be a sequence of subsets of X and let $E = \bigcup_{n=1}^{\infty} E_n$. If $\sum_{n=1}^{\infty} \mu^*(E_n) = \infty$, then $\nu(E) = \mu^*(E) \le \sum_{n=1}^{\infty} \mu^*(E_n) = \sum_{n=1}^{\infty} \nu(E_n)$ is trivially true. So, assume $\sum_{n=1}^{\infty} \mu^*(E_n) < \infty$ and let $\varepsilon > 0$. For each n pick a sequence $\{A_n^k\}$ of $\mathcal{F}$ with $E_n \subseteq \bigcup_{k=1}^{\infty} A_n^k$ and

$$\sum_{k=1}^{\infty} \mu(A_n^k) < \mu^*(E_n) + 2^{-n}\varepsilon = \nu(E_n) + 2^{-n}\varepsilon.$$

Clearly, $E \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} A_n^k$ holds, and so

$$\nu(E) = \mu^*(E) \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(A_n^k) < \sum_{n=1}^{\infty}\big[\nu(E_n) + 2^{-n}\varepsilon\big] = \sum_{n=1}^{\infty} \nu(E_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\nu(E) \le \sum_{n=1}^{\infty} \nu(E_n)$, and we are done.

Problem 14.12. *Consider an outer measure μ on a set X and let $\mathcal{A}$ be the collection of all measurable subsets of X of finite measure. That is, consider the family $\mathcal{A} = \{A \in \Lambda\colon\ \mu(A) < \infty\}$.*

a. *Show that $\mathcal{A}$ is a semiring.*
b. *Define a relation $\simeq$ on $\mathcal{A}$ by $A \simeq B$ if $\mu(A \Delta B) = 0$. Show that $\simeq$ is an equivalence relation on $\mathcal{A}$.*
c. *Let D denote the set of all equivalence classes of $\mathcal{A}$. For $A \in \mathcal{A}$ let $\dot{A}$ denote the equivalence class of A in D. Now, for $\dot{A}, \dot{B} \in D$ define $d(\dot{A}, \dot{B}) = \mu(A \Delta B)$. Show that d is well defined and that (D, d) is a complete metric space.*

Solution. Note that if A, B, and C are three arbitrary sets, then

$$A \Delta C \subseteq (A \Delta B) \cup (B \Delta C).$$

(a) Straightforward. (Note that in actuality $\mathcal{A}$ is a ring of sets.)
(b) If A, B, and C in $\mathcal{A}$ satisfy $A \simeq B$ and $B \simeq C$, then the relation

$$\mu(A \Delta C) \le \mu\big((A \Delta B) \cup (B \Delta C)\big) \le \mu(A \Delta B) + \mu(B \Delta C) = 0$$

shows that $A \simeq C$.
(c) If $A \simeq A_1$ and $B \simeq B_1$, then

$$\begin{aligned}\mu(A \Delta B) &\le \mu\big((A \Delta A_1) \cup (A_1 \Delta B_1) \cup (B_1 \Delta B)\big)\\ &\le \mu(A \Delta A_1) + \mu(A_1 \Delta B_1) + \mu(B_1 \Delta B) = \mu(A_1 \Delta B_1).\end{aligned}$$

Similarly, $\mu(A_1 \Delta B_1) \le \mu(A \Delta B)$, and so $\mu(A \Delta B) = \mu(A_1 \Delta B_1)$. This shows that $d(\dot{A}, \dot{B}) = \mu(A \Delta B)$ is well defined.

For the triangle inequality, note that

$$d(\dot{A}, \dot{B}) = \mu(A \Delta B) \le \mu(A \Delta C) + \mu(C \Delta B) = d(\dot{A}, \dot{C}) + d(\dot{C}, \dot{B}).$$

Thus, (D, d) is a metric space. What remains to be shown is that (D, d) is a complete metric space.

To this end, let $\{\dot{A}_n\}$ be a Cauchy sequence of D. By passing to a subsequence, we can assume that

$$d(\dot{A}_{n+1}, \dot{A}_n) = \mu(A_{n+1} \Delta A_n) < 2^{-n-1}$$

holds for each n. Set $A = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i \in \Lambda$. Now, let n be fixed and note that $A \subseteq \bigcup_{i=n}^{\infty} A_i = A_n \cup \left(\bigcup_{i=n}^{\infty} (A_{i+1} \setminus A_i)\right)$ holds. Thus,

$$\mu(A) \le \mu(A_n) + \sum_{i=n}^{\infty} \mu(A_{i+1} \setminus A_i) < \mu(A_n) + 2^{-n} < \infty,$$

and so $\dot{A} \in D$. Moreover, we have

$$\mu(A \setminus A_n) \le \mu\Big(\bigcup_{i=n}^{\infty} (A_{i+1} \setminus A_i)\Big) \le \sum_{i=n}^{\infty} \mu(A_{i+1} \setminus A_i) < 2^{-n}.$$

On the other hand, if $x \in A_n \setminus A$, then $x \in A_n$ and $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i$. Consequently, there exists some $k \ge n$ with $x \notin A_i$ for each $i \ge k$. This implies $A_n \setminus A \subseteq \bigcup_{i=n}^{\infty} (A_i \setminus A_{i+1})$, and so $\mu(A_n \setminus A) \le \sum_{i=n}^{\infty} \mu(A_i \setminus A_{i+1}) < 2^{-n}$ also holds. Therefore,

$$d(\dot{A}_n, \dot{A}) = \mu(A_n \Delta A) = \mu(A_n \setminus A) + \mu(A \setminus A_n) < 2^{1-n}$$

holds for each n. This shows that $\lim d(\dot{A}_n, \dot{A}) = 0$, and so (D, d) is a complete metric space. (For an alternate proof of this part, see Problem 31.3.)

15. THE OUTER MEASURE GENERATED BY A MEASURE

Problem 15.1. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let E be a measurable subset of X. Put $\mathcal{S}_E = \{E \cap A\colon A \in \mathcal{S}\}$, the restriction of $\mathcal{S}$ to E. Show that $(E, \mathcal{S}_E, \mu^*)$ is a measure space.*

Solution. Let E be a measurable subset of X and let $\{A_n\}$ be a sequence of Λ such that

a. $\{A_n \cap E\}$ is a pairwise disjoint sequence; and
b. there exists some $A \in \mathcal{S}$ such that $A \cap E = \bigcup_{n=1}^{\infty} A_n \cap E$.

Using the fact that $\mu^*\colon \Lambda \longrightarrow [0, \infty]$ is a measure, we see that

$$\mu^*(A \cap E) = \mu^*\Bigl(\bigcup_{n=1}^{\infty}(A_n \cap E)\Bigr) = \sum_{n=1}^{\infty} \mu^*(A \cap E),$$

and so μ^* is a measure when restricted to the semiring $\mathcal{S}_E$.

Problem 15.2. *Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that*

$$\mu^*(A) = \inf\{\mu^*(B)\colon\ B \text{ is a } \sigma\text{-set such that } A \subseteq B\}$$

holds for every subset A of X.

Solution. Let $\{A_n\} \subseteq \mathcal{S}$ and let $B = \bigcup_{n=1}^{\infty} A_n$. By Theorem 12.2(3), there exists a pairwise disjoint sequence $\{B_n\}$ of $\mathcal{S}$ such that $B = \bigcup_{n=1}^{\infty} B_n$. Thus, for $A \subseteq X$, there exists a sequence $\{A_n\}$ of $\mathcal{S}$ with $A \subseteq \bigcup_{n=1}^{\infty} A_n$ if and only if there exists a σ-set B with $A \subseteq B$. The desired equality now follows from the relation

$$\mu^*(A) \le \sum_{n=1}^{\infty} \mu(B_n) = \mu^*(B) \le \sum_{n=1}^{\infty} \mu(A_n).$$

Problem 15.3. *Show that every interval I of $\mathbf{R}$ is Lebesgue measurable and $\lambda^*(I) = |I|$ (=the length of I).*

Solution. In Example 15.5, we established that the intervals I of the form $[a, b]$ and $[a, \infty)$ are Lebesgue measurable and that $\lambda^*(I) = |I|$ holds for these cases. We shall consider the other cases separately. Assume $-\infty < a < b < \infty$.

a. $I = (a, b]$. Choose a sequence $\{x_n\}$ of real numbers with $x_n \downarrow a$ and $a < x_n < b$ for each n. Thus, by Example 15.5, we have

$$\lambda^*\bigl((a, b]\bigr) = \lim_{n\to\infty} \lambda^*\bigl([x_n, b]\bigr) = \lim_{n\to\infty} (b - x_n) = b - a = |I|.$$

b. $I = (a, b)$. Pick $a < x_n < b$ with $x_n \downarrow a$ and observe that $[x_n, b) \uparrow (a, b)$.

c. $I = (-\infty, a)$. Note that $[a-n, a) \uparrow (-\infty, a)$ and so

$$\lambda^*\big((-\infty, a)\big) = \lim_{n\to\infty} \lambda\big([a-n, a)\big) = \lim_{n\to\infty} n = \infty = |I|.$$

d. $I = (-\infty, a]$. Note that $(-\infty, a) \subseteq (-\infty, a]$ and so from the inequality

$$\infty = |(-\infty, a)| = \lambda^*\big((-\infty, a)\big) \le \lambda^*\big((-\infty, a]\big),$$

we see that $|I| = \lambda^*(I) = \infty$.

e. $I = (a, \infty)$. The conclusion follows immediately from the obvious inclusion $[a+1, \infty) \subseteq (a, \infty)$.

f. $I = (-\infty, \infty)$. Note that $[0, \infty) \subseteq (-\infty, \infty)$.

Problem 15.4. *Show that every countable subset of* $\mathbf{R}$ *has Lebesgue measure zero.*

Solution. Let $a \in \mathbf{R}$. Then, $\{a\} \subseteq [a-\varepsilon, a+\varepsilon)$ holds for each $\varepsilon > 0$ and so $\lambda^*(\{a\}) \le \lambda^*\big([a-\varepsilon, a+\varepsilon)\big) = 2\varepsilon$ for all $\varepsilon > 0$. Therefore, $\lambda^*(\{a\}) = 0$ holds for all $a \in \mathbf{R}$. If $A = \{a_1, a_2, \ldots\} = \bigcup_{n=1}^{\infty}\{a_n\}$ is a countable set, then note that $\lambda^*(A) \le \sum_{n=1}^{\infty} \lambda^*(\{a_n\}) = 0$ so that $\lambda^*(A) = 0$.

Problem 15.5. *For a subset* A *of* $\mathbf{R}$ *and real numbers* a *and* b*, define the set* $aA + b = \{ax + b\colon x \in A\}$. *Show that*

a. $\lambda^*(aA+b) = |a|\lambda^*(A)$, *and*

b. *if* A *is Lebesgue measurable, then so is* $aA+b$.

Solution. Let $A \subseteq \mathbf{R}$ and fix two real numbers a and b. Since $A \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n)$ holds if and only if $A + b \subseteq \bigcup_{n=1}^{\infty} [a_n + b, b_n + b)$ holds, it is easy to see that $\lambda^*(A+b) = \lambda^*(A)$. The identities

$$E \cap (b+A) = b + (E-b) \cap A \quad \text{and} \quad E \cap (b+A)^c = b + (E-b) \cap A^c$$

imply

$$\lambda^*\big(E \cap (b+A)\big) + \lambda^*\big(E \cap (b+A)^c\big) = \lambda^*\big((E-b) \cap A\big) + \lambda^*\big((E-b) \cap A^c\big),$$

which shows that A is measurable if and only if $b + A$ is measurable for each $b \in \mathbf{R}$.

Next, note that $\lambda^*\big(c(s,t)\big) = |c|\lambda^*\big((s,t)\big) = |c|(t-s)$ holds. On the other hand, since $A \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$ holds if and only if $aA \subseteq \bigcup_{n=1}^{\infty} a(a_n, b_n)$ holds for each

$a \in \mathbb{R}$ and since $\lambda^*\big([a_n, b_n)\big) = \lambda^*\big((a_n, b_n)\big)$, it follows that $\lambda^*(aA) = |a|\lambda^*(A)$ for each $a \in \mathbb{R}$. Now, the identities

$$E \cap aA = a\big((a^{-1}E) \cap A\big) \quad \text{and} \quad E \cap (aA)^c = a\big((a^{-1}E) \cap A^c\big)\ (a \neq 0),$$

imply

$$\lambda^*(E \cap aA) + \lambda^*\big(E \cap (aA)^c\big) = |a|\big[\lambda^*\big((a^{-1}E) \cap A\big) + \lambda^*\big((a^{-1}E) \cap A^c\big)\big],$$

which shows that A is measurable if and only if aA is measurable for each $a \in \mathbb{R}$.

Now, (a) and (b) follow from the preceding discussion.

Problem 15.6. *Let S be a semiring of subsets of a set X, and let $\mu: S \to [0, \infty]$ be a finitely additive measure that is not a measure. For each $A \subseteq X$ define (as usual)*

$$\mu^*(A) = \inf\Big\{\sum_{n=1}^{\infty} \mu(A_n) \colon \{A_n\} \subseteq S \text{ and } A \subseteq \bigcup_{n=1}^{\infty} A_n\Big\}.$$

Show by a counterexample that it is possible to have $\mu \neq \mu^*$ on S. Why does this not contradict Theorem 15.1?

Solution. Consider the finitely additive measure μ of Problem 13.7. Clearly, $\mu(\mathbb{N}) = \infty$. Since $\mathbb{N} = \bigcup_{n=1}^{\infty}\{n\} \in S$, we have $\mu^*(\mathbb{N}) \leq \sum_{n=1}^{\infty} \mu(\{n\}) = 0$, and so $0 = \mu^*(\mathbb{N}) < \mu(\mathbb{N}) = \infty$.

This conclusion does not contradict Theorem 15.1, since the σ-additivity of the measure was essential for its proof.

Problem 15.7. *Let E be an arbitrary measurable subset of a measure space (X, S, μ) and consider the measure space (E, S_E, ν), where $S_E = \{E \cap A\colon A \in S\}$ and $\nu(E \cap A) = \mu^*(E \cap A)$; see Problem 15.1. Establish the following properties regarding the measure space (E, S_E, ν):*

a. *The outer measure ν^* is the restriction of μ^* on E, i.e., $\nu^*(B) = \mu^*(B)$ for each $B \subseteq E$.*
b. *The ν-measurable sets of the measure space (E, S_E, ν) are precisely the sets of the form $E \cap A$ where A is a μ-measurable subset of X, i.e.,*

$$\Lambda_\nu = \{F \subseteq E\colon F \in \Lambda_\mu\}.$$

Solution. Let $(X, \mathcal{S}, \mu)$, E, and ν be as defined in the problem.

(a) Let B be an arbitrary subset of E. If $\{A_n\}$ is a sequence of $\mathcal{S}$ satisfying $B \subseteq \bigcup_{n=1}^{\infty} A_n$, then note that $B \subseteq \bigcup_{n=1}^{\infty} E \cap A_n$ and so

$$\nu^*(B) \le \sum_{n=1}^{\infty} \nu(E \cap A_n) = \sum_{n=1}^{\infty} \mu^*(E \cap A_n) \le \sum_{n=1}^{\infty} \mu(A_n).$$

This implies $\nu^*(B) \le \mu^*(B)$. On the other hand, if $\{A_n\}$ is a sequence of $\mathcal{S}$ satisfying $B \subseteq \bigcup_{n=1}^{\infty} E \cap A_n$, then we have

$$\mu^*(B) \le \sum_{n=1}^{\infty} \mu^*(E \cap A_n) = \sum_{n=1}^{\infty} \nu(E \cap A_n).$$

Thus, $\mu^*(B) \le \nu^*(B)$ also holds, and so $\nu^*(B) = \mu^*(B)$ for each subset B of E.

(b) Let F be a subset of E. Assume first that F is ν-measurable. If $A \in \mathcal{S}$, then note that

$$\begin{aligned}
\mu(A) &= \mu^*(A \cap E) + \mu^*\big(A \cap (X \setminus E)\big) \\
&= \nu^*(A \cap E) + \mu^*\big(A \cap (X \setminus E)\big) \\
&= \nu^*\big((A \cap E) \cap F\big) + \nu^*\big((A \cap E) \cap (E \setminus F)\big) + \mu^*\big(A \cap (X \setminus E)\big) \\
&\ge \mu^*(A \cap F) + \mu^*\big([A \cap (E \setminus F)] \cup [A \cap (X \setminus E)]\big) \\
&= \mu^*(A \cap F) + \mu^*\big(A \cap (X \setminus F)\big),
\end{aligned}$$

which shows that F is μ-measurable.

For the converse, assume that F is μ-measurable. If A is an arbitrary subset of E, then note that

$$\begin{aligned}
\nu^*(A) = \mu^*(A) &= \mu^*(A \cap F) + \mu^*\big(A \cap (X \setminus F)\big) \\
&= \mu^*(A \cap F) + \mu^*\big(A \cap (E \setminus F)\big) \\
&= \nu^*(A \cap F) + \nu^*\big(A \cap (E \setminus F)\big),
\end{aligned}$$

which means that F is also ν-measurable.

Problem 15.8. *Show that a subset E of a measure space $(X, \mathcal{S}, \mu)$ is measurable if and only if for each $\epsilon > 0$ there exists a measurable set A_ϵ and two subsets B_ϵ and C_ϵ satisfying*

$$E = (A_\epsilon \cup B_\epsilon) \setminus C_\epsilon, \quad \mu^*(B_\epsilon) < \epsilon, \quad \textit{and} \quad \mu^*(C_\epsilon) < \epsilon.$$

Solution. Let E be a subset of a measure space (X, S, μ). If E is a measurable set and $\varepsilon > 0$ is given, then let $A_\varepsilon = E$ and $B_\varepsilon = C_\varepsilon = \emptyset$, and note that these sets satisfy the desired properties.

For the converse, assume that for each $\varepsilon > 0$ there exist a measurable set A_ε and subsets B_ε and C_ε satisfying

$$E = (A_\varepsilon \cup B_\varepsilon) \setminus C_\varepsilon, \quad \mu^*(B_\varepsilon) < \varepsilon, \quad \text{and} \quad \mu^*(C_\varepsilon) < \varepsilon. \tag{$\star$}$$

Replacing C_ε by $(A_\varepsilon \cup B_\varepsilon) \cap C_\varepsilon$, we can assume that C_ε is a subset of $A_\varepsilon \cup B_\varepsilon$. From $(\star)$, we see that

$$E \cup C_\varepsilon = A_\varepsilon \cup B_\varepsilon. \tag{$\star\star$}$$

Now, by Theorem 15.11, there exists a measurable set D_ε such that $B_\varepsilon \subseteq D_\varepsilon$ and $\mu^*(D_\varepsilon) = \mu^*(B_\varepsilon)$. Using $(\star\star)$, we get

$$E \cup C_\varepsilon \cup (D_\varepsilon \setminus B_\varepsilon) = A_\varepsilon \cup B_\varepsilon \cup (D_\varepsilon \setminus B_\varepsilon) = A_\varepsilon \cup D_\varepsilon.$$

Clearly, $A_\varepsilon \cup D_\varepsilon$ is a measurable set and

$$\mu^*(C_\varepsilon \cup (D_\varepsilon \setminus B_\varepsilon)) \le \mu^*(C_\varepsilon) + \mu^*(D_\varepsilon) < 2\varepsilon.$$

In other words, the preceding show that for each $\varepsilon > 0$ there exist a measurable set F_ε and a subset G_ε such that

$$E \cup G_\varepsilon = F_\varepsilon \quad \text{and} \quad \mu^*(G_\varepsilon) < \varepsilon.$$

Now, for each n pick a measurable set F_n and a subset G_n with $\mu^*(G_n) < \frac{1}{n}$ and $E \cup G_n = F_n$. Clearly, the set $F = \bigcap_{n=1}^{\infty} F_n$ is measurable. Also, the set $G = \bigcap_{n=1}^{\infty} G_n$ is a null set—and hence $G \setminus E$ is also measurable. In view of

$$E \cup G = \bigcap_{n=1}^{\infty} (E \cup G_n) = \bigcap_{n=1}^{\infty} F_n = F,$$

we see that $E \cup G$ is a measurable set. Finally, the measurability of E follows immediately from the identity

$$E = (E \cup G) \setminus (G \setminus E) = F \setminus (G \setminus E).$$

Problem 15.9. *Let (X, S, μ) be a measure space, and let A be a subset of X. Show that if there exists a measurable subset E of X such that $A \subseteq E$, $\mu^*(E) < \infty$, and $\mu^*(E) = \mu^*(A) + \mu^*(E \setminus A)$, then A is measurable.*

Solution. By Problem 15.7, we know that the outer measure generated by the measure space $(E, \mathcal{S}_E, \mu^*)$ coincides with μ^* and the σ-algebra of all measurable sets of the measure space $(E, \mathcal{S}_E, \mu^*)$ is $\{A \in \Lambda_\mu\colon A \subseteq E\}$.

Now, to complete the proof, assume $E \in \Lambda_\mu$ and that a subset A of E satisfies $\mu^*(A) + \mu^*(E \setminus A) = \mu^*(E)$. If $\mu^*(E) < \infty$ holds, then it follows from Theorem 15.8 that A is a measurable set for $(E, \mathcal{S}_E, \mu^*)$. Thus, by the preceding discussion, $A \in \Lambda_\mu$.

Problem 15.10. *Let A be a subset of $\mathbf{R}$ with $\lambda^*(A) > 0$. Show that there exists a nonmeasurable subset B of $\mathbf{R}$ such that $B \subseteq A$.*

Solution. If A is nonmeasurable, then there is nothing to prove. So, assume that A is measurable. Since some $[n, n+1] \cap A$ must have nonzero measure (why?), by translating appropriately (and using Problem 15.5), we can also assume that $A \subseteq [0, 1]$.

As in Example 15.13 define an equivalence relation $\simeq$ on A by saying that $x \simeq y$ whenever $x - y$ is a rational number. By the Axiom of Choice, there exists a subset B of A containing precisely one member from each equivalence class. Let $\{r_1, r_2, \ldots\}$ be an enumeration of the rationals of $[-1, 1]$ and let $B_n = r_n + B$. Then:

a. The sequence $\{B_n\}$ is pairwise disjoint;
b. $\lambda^*(B_n) = \lambda^*(B)$ holds (by Problem 15.5) for each n; and
c. $A \subseteq \bigcup_{n=1}^{\infty} B_n \subseteq [-1, 2]$.

Now, note that if B is a measurable set, then each B_n is likewise a measurable set (see Problem 15.5 again). Thus, from (c), it follows that

$$\begin{aligned} 0 < \lambda^*(A) \leq \lambda^*\Big(\bigcup_{i=1}^{\infty} B_i\Big) &= \sum_{i=1}^{\infty} \lambda^*(B_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \lambda^*(B_i) = \lim_{n\to\infty} n\lambda^*(B) \leq 3, \end{aligned}$$

which is impossible. Therefore, B is a nonmeasurable subset of A.

Problem 15.11. *Give an example of a disjoint sequence $\{E_n\}$ of subsets of some measure space $(X, \mathcal{S}, \mu)$ such that*

$$\mu^*\Big(\bigcup_{n=1}^{\infty} E_n\Big) < \sum_{n=1}^{\infty} \mu^*(E_n).$$

Solution. Let E_n be the disjoint sequence of nonmeasurable sets described in Example 15.13, where $E_n = r_n + E$. Since E is a nonmeasurable set, $\lambda^*(E) > 0$ holds, and so $\lambda^*(E_n) = \lambda^*(E) > 0$. In particular, $\sum_{n=1}^{\infty} \lambda^*(E_n) = \infty$. On the other hand, $\bigcup_{n=1}^{\infty} E_n \subseteq [-1, 2]$ implies $\lambda^*\big(\bigcup_{n=1}^{\infty} E_n\big) \leq 3 < \infty$.

Problem 15.12. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $\{A_n\}$ be a sequence of subsets of X such that $A_n \subseteq A_{n+1}$ holds for all n. If $A = \bigcup_{n=1}^{\infty} A_n$, then show that $\mu^*(A_n) \uparrow \mu^*(A)$.*

Solution. Choose some $E \in \Lambda$ with $A \subseteq E$ and $\mu^*(A) = \mu^*(E)$. (This is possible by Theorem 15.11.) By the same theorem, for each n there exists some $E_n \in \Lambda$ with $A_n \subseteq E_n \subseteq E$ and $\mu^*(A_n) = \mu^*(E_n)$. Now, for each n put $F_n = \bigcap_{k=n}^{\infty} E_k \in \Lambda$, and then let $F = \bigcup_{n=1}^{\infty} F_n \in \Lambda$. Then, we have:

a. $A_n \subseteq F_n$ and $\mu^*(A_n) = \mu^*(F_n)$ for each n; and
b. $F_n \uparrow F$ and $\mu^*(A) = \mu^*(F)$.

By Theorem 15.4, it follows that

$$\mu^*(A_n) = \mu^*(F_n) \uparrow \mu^*(F) = \mu^*(A).$$

Problem 15.13. *For subsets of a measure space $(X, \mathcal{S}, \mu)$ let us define the following almost everywhere (a.e.) relations:*

a. *$A \subseteq B$ a.e. if $\mu^*(A \setminus B) = 0$;*
b. *$A = B$ a.e. if $\mu^*(A \Delta B) = 0$;*
c. *$A_n \uparrow A$ a.e. if $A_n \subseteq A_{n+1}$ a.e. for all n and $A = \bigcup_{n=1}^{\infty} A_n$ a.e. (The meaning of $A_n \downarrow A$ a.e. is similar.)*

Generalize Theorem 15.4 by establishing the following properties for a sequence $\{E_n\}$ of measurable sets:

i. *If $E_n \uparrow E$ a.e., then $\mu^*(E_n) \uparrow \mu^*(E)$.*
ii. *If $E_n \downarrow E$ a.e. and $\mu^*(E_k) < \infty$ for some k, then $\mu^*(E_n) \downarrow \mu^*(E)$.*

Is (i) true without assuming measurability for the sets E_n?

Solution. (i) Assume that $\{E_n\}$ is a sequence of measurable sets such that $E_n \uparrow E$ a.e. holds. Let

$$A = \Big[\Big(\bigcup_{n=1}^{\infty} E_n\Big) \Delta E\Big] \cup \Big[\bigcup_{n=1}^{\infty} (E_n \setminus E_{n+1})\Big]$$

and note that $\mu^*(A) = 0$. Now, define $F = E \cup A$ and $F_n = E_n \cup A$ for each n.

Clearly, $\mu^*(E) = \mu^*(F)$ and $\mu^*(E_n) = \mu^*(F_n)$ for each n (see Problem 14.2) and $F_n \uparrow F$.

Now, apply Theorem 15.4(1) to get

$$\mu^*(E_n) = \mu^*(F_n) \uparrow \mu^*(F) = \mu^*(E).$$

(ii) Assume that $\{E_n\}$ is a sequence of measurable sets such that $E_n \downarrow E$ a.e. and $\mu^*(E_k) < \infty$ holds for some k. Define $B = \left[\left(\bigcap_{n=1}^{\infty} E_n\right)\Delta E\right] \cup \left[\bigcup_{n=1}^{\infty}(E_{n+1} \setminus E_n)\right]$. Clearly, $\mu^*(B) = 0$. Now, apply Theorem 15.4(2) to $E_n \cup E \cup B \downarrow E \cup B$.

Statement (i) is also true without assuming measurability for the E_n. This follows from the arguments of (i) previously and Problem 15.12.

Problem 15.14. *Give an example of a sequence $\{E_n\}$ of measurable sets of some measure space $(X, \mathcal{S}, \mu)$ such that $E_{n+1} \subseteq E_n$ holds for all n and*

$$\lim_{n\to\infty} \mu^*(E_n) > \mu^*\Big(\bigcap_{n=1}^{\infty} E_n\Big).$$

Solution. Consider $\mathbb{R}$ with the Lebesgue measure, and let $E_n = (n, \infty)$ for each n. Then, $E_n \downarrow \emptyset$ holds, while $\lambda^*(E_n) = \infty$ for each n.

Problem 15.15. *For a sequence $\{A_n\}$ of subsets of a set X define*

$$\liminf A_n = \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} A_i \quad \textit{and} \quad \limsup A_n = \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i.$$

Now, let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{E_n\}$ be the sequence of measurable sets. Show the following:

a. $\mu^*(\liminf E_n) \le \liminf \mu^*(E_n)$.
b. *If $\mu^*(\bigcup_{n=1}^{\infty} E_n) < \infty$, then $\mu^*(\limsup E_n) \ge \limsup \mu^*(E_n)$.*

Solution. (a) Note that $\bigcap_{i=n}^{\infty} E_i \uparrow \liminf E_n$ and $\bigcap_{i=n}^{\infty} E_i \subseteq E_n$ holds for each n. By Theorem 15.4(1), we get

$$\mu^*(\liminf E_n) = \lim_{n\to\infty} \mu^*\Big(\bigcap_{i=n}^{\infty} E_i\Big) \le \liminf \mu^*(E_n).$$

(b) Use similar arguments and Theorem 15.4(2).

Problem 15.16. *Give an example of a sequence $\{A_n\}$ of subsets of some measure space $(X, \mathcal{S}, \mu)$ such that $A_{n+1} \subseteq A_n$ holds for each n, $\mu^*(A_1) < \infty$, and*

$$\lim_{n\to\infty} \mu^*(A_n) > \mu^*\Big(\bigcap_{n=1}^{\infty} A_n\Big).$$

Solution. Let $A_n = \bigcup_{i=n}^{\infty} E_i$, where $\{E_n\}$ is the sequence of Example 15.13. Note that $A_n \downarrow \emptyset$ holds. Indeed, if $x \in A_n$, then $x \in E_k$ for some $k \geq n$. Since $E_i \cap E_j = \emptyset$ whenever $i \neq j$, it follows that $x \notin A_{k+1}$ so that $A_n \downarrow \emptyset$ holds. Now, observe that $\lambda^*(A_n) \geq \lambda^*(E_n) = \lambda^*(E) > 0$ holds for all n.

Problem 15.17. *Let $(X, \mathcal{S}_1, \mu_1)$ and $(X, \mathcal{S}_2, \mu_2)$ be two measure spaces. Show that μ_1 and μ_2 generate the same outer measure on X if and only if $\mu_1 = \mu_2^*$ on $\mathcal{S}_1$ and $\mu_2 = \mu_1^*$ on $\mathcal{S}_2$ both hold.*

Solution. If μ_1 and μ_2 generate the same outer measure, then clearly $\mu_1 = \mu_2^*$ on $\mathcal{S}_1$ and $\mu_2 = \mu_1^*$ on $\mathcal{S}_2$ both hold.

For the converse, assume that $\mu_1 = \mu_2^*$ on $\mathcal{S}_1$ and $\mu_2 = \mu_1^*$ on $\mathcal{S}_2$ both hold. Let $A \subseteq X$. If $\mu_2^*(A) = \infty$, then $\mu_1^*(A) \leq \mu_2^*(A)$ holds. If $\mu_2^*(A) < \infty$, then given $\varepsilon > 0$ there exists a sequence $\{A_n\}$ of $\mathcal{S}_2$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty} \mu_2(A_n) < \mu_2^*(A) + \varepsilon$. Thus,

$$\mu_1^*(A) \leq \sum_{n=1}^{\infty} \mu_1^*(A_n) = \sum_{n=1}^{\infty} \mu_2(A_n) < \mu_2^*(A) + \varepsilon$$

holds for all $\varepsilon > 0$, and so $\mu_1^*(A) \leq \mu_2^*(A)$.

Similarly, $\mu_1^*(A) \geq \mu_2^*(A)$ holds, and therefore $\mu_1^*(A) = \mu_2^*(A)$ holds for all $A \subseteq X$.

Problem 15.18. *Let $(X, \mathcal{S}, \mu)$ be a measure space. A measurable set A is called an* **atom** *if $\mu^*(A) > 0$ and for every measurable subset E of A we have either $\mu^*(E) = 0$ or $\mu^*(A \setminus E) = 0$. If $(X, \mathcal{S}, \mu)$ does not have any atoms, then it is called a* **nonatomic measure space**.

a. *Find the atoms of:*
   i. *the counting measure, and*
   ii. *the Dirac measure based at a point a.*

b. *Show that the real line with the Lebesgue measure is a nonatomic measure space.*

Solution. a. (i) The atoms of the counting measure on a set are precisely the one-point sets.

(ii) The atoms of the Dirac measure based at a point a are precisely the sets containing the point a.
b. Let $A \subseteq \mathbf{R}$ be measurable with $\lambda^*(A) > 0$. Pick some integer n such that $\lambda^*([n, n+1] \cap A) = \delta > 0$. Subdivide $[n, n+1]$ into a finite number of subintervals all of the same length less than δ. For one of them, say I, we must have

$$\lambda^*\big([n, n+1] \cap A \cap I\big) > 0.$$

Now, note that the set $E = [n, n+1] \cap A \cap I \subseteq A$ is measurable and satisfies $0 < \lambda^*(E) < \delta \leq \lambda^*(A)$. This shows that A is not an atom, and hence $\mathbf{R}$ with the Lebesgue measure is nonatomic. (For more about this problem, see Problem 18.19.)

Problem 15.19. *This exercise presents an example of a measure that has infinitely many extensions to a measure on the σ-algebra generated by $\mathcal{S}$. Fix a proper nonempty subset A of a set X (i.e., A $\neq$ X) and consider the collection of subsets $\mathcal{S} = \{\emptyset, A\}$.*

a. *Show that $\mathcal{S}$ is a semiring.*
b. *Show that the set function $\mu: \mathcal{S} \to [0, \infty]$ defined by $\mu(\emptyset) = 0$ and $\mu(A) = 1$ is a measure.*
c. *Describe the Carathéodory extension μ^* of μ.*
d. *Determine the σ-algebra of measurable sets Λ_μ.*
e. *Show that μ has uncountably many extensions to a measure on the σ-algebra generated by $\mathcal{S}$. Why doesn't this contradict Theorem 15.10?*

Solution. The validity of (a) and (b) should be obvious.
(c) The Carathéodory extension of μ is given by

$$\mu^*(B) = \begin{cases} 0 & \text{if } B = \emptyset; \\ 1 & \text{if } B \neq \emptyset \text{ and } B \subseteq A; \\ \infty & \text{if } B \nsubseteq A. \end{cases}$$

(d) The σ-algebra generated by $\mathcal{S}$ is

$$\mathcal{A} = \{\emptyset, A, A^c, X\}.$$

(e) If a is any non-negative extended real number, then the set function $\nu: \mathcal{A} \to [0, \infty]$, defined by

$$\nu(\emptyset) = 0, \quad \nu(A) = 1, \quad \nu(A^c) = a, \quad \text{and} \quad \nu(X) = 1 + a,$$

is a measure which is an extension of μ to all of $\mathcal{A}$. This shows that there are uncountably many extensions of μ to the σ-algebra generated by $\mathcal{S}$.

The latter conclusion does not contradict Theorem 15.10 because μ is not a σ-finite measure.

16. MEASURABLE FUNCTIONS

Problem 16.1. *Let $(X, \mathcal{S}, \mu)$ be a measure space. For a function $f: X \to \mathbf{R}$ show that the following statements are equivalent:*

a. *f is a measurable function.*
b. *$f^{-1}((-\infty, a))$ is measurable for each $a \in \mathbf{R}$.*
c. *$f^{-1}((a, \infty))$ is measurable for each $a \in \mathbf{R}$.*

Solution. (a)$\Longrightarrow$(b) Note that $f^{-1}\big((-\infty, a)\big)$ is a measurable set simply because the interval $(-\infty, a)$ is an open set.
(b)$\Longrightarrow$(c) Observe that the identity

$$f^{-1}\big((-\infty, a]\big) = \bigcap_{n=1}^{\infty} f^{-1}\big((-\infty, a + \tfrac{1}{n})\big)$$

implies that $f^{-1}\big((-\infty, a]\big)$ is a measurable set for each $a \in \mathbf{R}$. Consequently, the set $f^{-1}\big((a, \infty)\big) = X \setminus f^{-1}\big((-\infty, a]\big)$ is also measurable for each $a \in \mathbf{R}$.
(c)$\Longrightarrow$(a) Clearly, $f^{-1}\big((-\infty, a]\big) = X \setminus f^{-1}\big((a, \infty)\big)$ is measurable for each $a \in \mathbf{R}$. Thus, by condition (5) of Theorem 16.2, the function f is measurable.

Problem 16.2. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let A be a dense subset of $\mathbf{R}$. Show that a function $f: X \to \mathbf{R}$ is measurable if and only if the set $\{x \in X: f(x) \geq a\}$ is measurable for each $a \in A$.*

Solution. Only the "if" part needs proof. Let $a \in \mathbf{R}$. Since A is dense in $\mathbf{R}$, there exists a sequence $\{a_n\}$ of A with $a_n < a$ for each n and $a_n \uparrow a$. Now, note that the identity

$$f^{-1}\big([a, \infty)\big) = \bigcap_{n=1}^{\infty} f^{-1}\big([a_n, \infty)\big)$$

shows that the set $f^{-1}\big([a, \infty)\big)$ is measurable. Therefore, by Theorem 16.2, the function f is measurable.

Problem 16.3. *Give an example of a nonmeasurable function f such that $|f|$ is a measurable function and $f^{-1}(\{a\})$ is a measurable set for each $a \in \mathbf{R}$.*

Solution. Take a non-Lebesgue measurable subset E of $[0, 1]$ and consider the function $f: [0, 1] \longrightarrow \mathbf{R}$ defined by

$$f(x) = \begin{cases} x, & \text{if } x \in E; \\ -x, & \text{if } x \in [0, 1] \setminus E. \end{cases}$$

It is straightforward to verify that the function f satisfies the desired properties.

Problem 16.4. *Show that if $f: \mathbf{R} \to \mathbf{R}$ is continuous a.e., then f is a Lebesgue measurable function.*

Solution. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be a function that is continuous almost everywhere. Put $E = \{x \in \mathbf{R}: f \text{ is continuous at } x\}$ and note that $\lambda^*(\mathbf{R} \setminus E) = 0$. Hence, $\mathbf{R} \setminus E$ and E are both measurable sets.

Now, let $\mathcal{O}$ be an arbitrary open subset of $\mathbf{R}$. Clearly, the set $f^{-1}(\mathcal{O}) \cap (\mathbf{R} \setminus E)$ (as a null set) is measurable. Since f restricted to E is continuous, $f^{-1}(\mathcal{O}) \cap E$ is an open set in E, and consequently there exists an open subset V of $\mathbf{R}$ such that $f^{-1}(\mathcal{O}) \cap E = V \cap E$. In particular, note that $f^{-1}(\mathcal{O}) \cap E$ is a measurable set. Therefore,

$$f^{-1}(\mathcal{O}) = \left[f^{-1}(\mathcal{O}) \cap E\right] \cup \left[f^{-1}(\mathcal{O}) \cap (\mathbf{R} \setminus E)\right]$$

is likewise measurable, so that f is a measurable function.

Problem 16.5. *Let $f: \mathbf{R} \to \mathbf{R}$ be a differentiable function. Show that f' is Lebesgue measurable.*

Solution. For each n define

$$g_n(x) = n\left[f(x + \tfrac{1}{n}) - f(x)\right] = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}},$$

and note that each g_n is measurable (since it is continuous). In view of $g_n(x) \to f'(x)$ for each $x \in \mathbf{R}$, it follows from Theorem 16.6 that f' is a measurable function.

Problem 16.6. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f: X \to \mathbf{R}$ be a measurable function. Show that:*

a. *$|f|^p$ is a measurable function for all $p \geq 0$, and*
b. *if $f(x) \neq 0$ for each $x \in X$, then $1/f$ is a measurable function.*

Solution. Let $f: X \longrightarrow \mathbf{R}$ be a measurable function.

(a) Assume $p > 0$. By Theorem 16.5, $|f|$ is measurable. The conclusion now follows from the identities $\{x \in X: |f|^p(x) \geq a\} = X$ if $a \leq 0$, and $\{x \in X: |f|^p(x) \geq a\} = \{x \in X: |f(x)| \geq a^{\frac{1}{p}}\}$ if $a > 0$.

(b) Assume that $f(x) \neq 0$ holds for each $x \in X$. Note that

$$\{x \in X: \tfrac{1}{f}(x) > 0\} = \{x \in X: f(x) > 0\},$$

$$\{x \in X: \tfrac{1}{f}(x) > a\} = \{x \in X: f(x) < \tfrac{1}{a}\} \text{ if } a > 0, \text{ and}$$

$$\{x \in X: \tfrac{1}{f}(x) > a\} = \{x \in X: f(x) < \tfrac{1}{a}\} \cup \{x \in X: f(x) > 0\} \text{ if } a < 0.$$

The preceding identities guarantee that $\frac{1}{f}$ is measurable.

Problem 16.7. *Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measure space $(X, \mathcal{S}, \mu)$. Then show that the sets*

a. $A = \{x \in X: f_n(x) \to \infty\}$,
b. $B = \{x \in X: f_n(x) \to -\infty\}$, *and*
c. $C = \{x \in X: \lim f_n(x)$ *exists in* $\mathbf{R}\}$

are all measurable.

Solution. (a) For each m and k let $A_{m,k} = \{x \in X: f_n(x) \geq k \text{ for all } n \geq m\}$. From $A_{m,k} = \bigcap_{n=m}^{\infty}\{x \in X: f_n(x) \geq k\}$, we see that $A_{m,k} \in \Lambda_\mu$ for each m, k. Now, note that $A = \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty} A_{m,k}$.

(b) Put $B_{m,k} = \{x \in X: f_n(x) \leq -k \text{ for each } n \geq m\}$ and note that $B = \bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty} B_{m,k}$.

(c) Let $Y = X \setminus (A \cup B)$ and consider the measure space $(Y, \mathcal{S}_Y, \mu^*)$. Also, consider all functions restricted to Y. In view of Problem 15.7, all functions are measurable with respect to this space. By Theorem 16.6, both functions $\liminf f_n$ and $\limsup f_n$ are measurable. The conclusion now follows from Theorem 16.4(c) by observing that

$$\begin{aligned} C &= \{x \in X: \lim f_n(x) \text{ exists in } \mathbf{R}\} \\ &= \{x \in Y: \limsup f_n(x) = \liminf f_n(x)\}. \end{aligned}$$

Problem 16.8. *Let $(X, \mathcal{S}, \mu)$ be a measure space. Assume that $f: X \to \mathbf{R}$ is a measurable function and $g: \mathbf{R} \to \mathbf{R}$ is a continuous function. Show that $g \circ f$ is a measurable function.*

Solution. Consider the functions $X \xrightarrow{f} \mathbf{R} \xrightarrow{g} \mathbf{R}$ with f measurable and g continuous, and let $\mathcal{O}$ be an open subset of $\mathbf{R}$. Since g is continuous, we know

that $g^{-1}(\mathcal{O})$ is an open set, and the conclusion follows from the identity

$$(g \circ f)^{-1}(\mathcal{O}) = f^{-1}\big(g^{-1}(\mathcal{O})\big).$$

Problem 16.9. *Let $\mathcal{F}$ be a nonempty family of continuous real-valued functions defined on $\mathbf{R}$. Assume that there exists a function $g\colon \mathbf{R} \to \mathbf{R}$ such that $f(x) \le g(x)$ for each $x \in \mathbf{R}$ and all $f \in \mathcal{F}$. Show that the supremum function $h\colon \mathbf{R} \to \mathbf{R}$, defined by $h(x) = \sup\{f(x)\colon f \in \mathcal{F}\}$, is (Lebesgue) measurable.*

Solution. We shall show that $h^{-1}\big((a,\infty)\big)$ is an open set for each $a \in \mathbf{R}$ (and hence, a Lebesgue measurable set).

To see this, let $a \in \mathbf{R}$ and fix $x_0 \in h^{-1}\big((a,\infty)\big)$, i.e., $h(x_0) > a$. So, there exists some $f \in \mathcal{F}$ such that $f(x_0) > a$. Since f is a continuous function, there exists some neighborhood V of x_0 such that $f(x) > a$ for each $x \in V$. This implies $h(x) \ge f(x) > a$ for each $x \in V$, and so $V \subseteq h^{-1}\big((a,\infty)\big)$. This shows that x_0 is an interior point of $h^{-1}\big((a,\infty)\big)$ and consequently, $h^{-1}\big((a,\infty)\big)$ is an open set.

Note: A real-valued function $f\colon X \to \mathbf{R}$ defined on a topological space X is said to be *lower semicontinuous* if $f^{-1}\big((a,\infty)\big)$ is an open set for each $a \in \mathbf{R}$. The preceding arguments show that we have proven the following result: *The pointwise supremum of a family of lower semicontinuous functions is likewise lower semicontinuous.*

Problem 16.10. *Show that if $f\colon X \to \mathbf{R}$ is a measurable function, then either f is constant almost everywhere or else (exclusively) there exists a constant c such that*

$$\mu^*(\{x \in X\colon\ f(x) > c\}) > 0 \quad \textit{and} \quad \mu^*(\{x \in X\colon\ f(x) < c\}) > 0.$$

Solution. Let $f\colon X \to \mathbf{R}$ be a measurable function which is not a constant almost everywhere. Assume first that $f(x) \ge 0$ holds for each $x \in X$ and let

$$c_0 = \sup\big\{c \in \mathbf{R}\colon\ \mu^*(\{x \in X\colon\ f(x) \le c\} = 0\big\}.$$

Clearly, $0 \le c_0 < \infty$ and $\mu^*\big(\{x \in X\colon\ f(x) < c_0\}\big) = 0$. Since f is not constant almost everywhere, there exists some $c > c_0$ such that $\mu^*\big(\{x \in X\colon\ f(x) > c\}\big) > 0$. Now, if k satisfies $c_0 < k < c$, then by the definition of c_0 we have $\mu^*\big(\{x \in X\colon\ f(x) < c\}\big) \ge \mu^*\big(\{x \in X\colon\ f(x) \le k\}\big) > 0$, and the desired conclusion is established in this case.

In the general case, either f^+ or f^- is not equal to a constant almost everywhere. We consider the case where f^+ is not equal to a constant almost everywhere (the other case can be treated in a similar fashion). By the preceding case, there exists $c > 0$ with $\mu^*\big(\{x \in X\colon\ f^+(x) > c\}\big) > 0$ and $\mu^*\big(\{x \in X\colon\ f^+(x) < c\}\big) > 0$.

To finish the proof, notice that

$$\{x \in X\colon\ f^+(x) > c\} = \{x \in X\colon\ f(x) > c\}$$

and $\{x \in X\colon\ f^+(x) < c\} = \{x \in X\colon\ f(x) < c\}$.

17. SIMPLE AND STEP FUNCTIONS

Problem 17.1. *For subsets A and B of a set X, establish the following statements:*

1. $\chi_\emptyset = 0$ *and* $\chi_X = 1$.
2. $A \subseteq B$ *if and only if* $\chi_A \le \chi_B$.
3. $\chi_{A\cap B} = \chi_A \cdot \chi_B = \chi_A \wedge \chi_B$.
4. $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} = \chi_A \vee \chi_B$.
5. $\chi_{A\setminus B} = \chi_A - \chi_{A\cap B}$.
6. *If* $A = \bigcup_{n=1}^\infty A_n$ *and* $\{A_n\}$ *is a pairwise disjoint sequence of subsets of* X, *then* $\chi_A = \sum_{n=1}^\infty \chi_{A_n}$.
7. $\chi_{A\times B} = \chi_A \cdot \chi_B$. *(Here the set B can be considered to be a subset of some other set Y.)*

Solution. The proofs of the statements are straightforward. To indicate how one can prove them, we shall establish the validity of statements (3) and (7).
(3) We have

$$\begin{aligned}
\chi_{A\cap B}(x) &= \begin{cases} 1, & \text{if } x \in A\cap B \\ 0, & \text{if } x \notin A\cap B \end{cases} \\
&= \begin{cases} 1, & \text{if } x \in A \text{ and } x \in B \\ 0, & \text{if } x \notin A \\ 0, & \text{if } x \notin B \end{cases} \\
&= \chi_A(x)\cdot\chi_B(x) \\
&= \chi_A \cdot \chi_B(x) \\
&= \min\{\chi_A(x), \chi_B(x)\}.
\end{aligned}$$

(7) Note that

$$\begin{aligned}
\chi_{A\times B}(x,y) &= \begin{cases} 1, & \text{if } (x,y) \in A\times B \\ 0, & \text{if } (x,y) \notin A\times B \end{cases} \\
&= \begin{cases} 1, & \text{if } x \in A \text{ and } y \in B \\ 0, & \text{if } x \notin A \\ 0, & \text{if } y \notin B \end{cases} \\
&= \chi_A(x)\cdot\chi_B(y).
\end{aligned}$$

Problem 17.2. *Let ϕ be a step function and ψ a simple function such that $0 \le \psi \le \phi$ a.e. Show that ψ is a step function.*

Solution. Let $E = \{x: 0 \le \psi(x) \le \phi(x)\}$, and observe that $\mu^*(X \setminus E) = 0$. If $F = \{x \in X: \phi(x) > 0\}$, then the measurable set $A = (X \setminus E) \cup F$ satisfies $\mu^*(A) < \infty$, and $\psi(x) = 0$ for each $x \in X \setminus A$.

Problem 17.3. *Show that if $(X, \mathcal{S}, \mu)$ is a finite measure space, then every simple function is a step function.*

Solution. If ϕ is a simple function and $E = \{x \in X: \phi(x) \neq 0\}$, then note that $\mu^*(E) \le \mu^*(X) < \infty$ holds.

Problem 17.4. *Give an alternate proof of the linearity of the integral (Theorem 17.2) based on Problem 12.14.*

Solution. The linearity follows immediately from the following property.

- *If ϕ is a step function and $\phi = \sum_{j=1}^m b_j \chi_{B_j}$ is an arbitrary representation of ϕ, then*

$$I(\phi) = \sum_{j=1}^m b_j \mu^*(B_j).$$

We shall establish the preceding property below.

To this end, let $\phi = \sum_{i=1}^n a_i \chi_{A_i}$ be the standard representation of ϕ. Assume first that the B_j are pairwise disjoint. Since neither the function ϕ nor the sum $\sum_{j=1}^m b_j \mu^*(B_j)$ changes by deleting the terms with $b_j = 0$, we can assume that $b_j \neq 0$ for each j. In such a case, we have $\bigcup_{i=1}^n A_i = \bigcup_{j=1}^m B_j$. Moreover, note that $a_i \mu^*(A_i \cap B_j) = b_j \mu^*(A_i \cap B_j)$ for all i and j. Indeed, if $A_i \cap B_j = \emptyset$ the equality is obvious and if $x \in A_i \cap B_j$, then $a_i = b_j = \phi(x)$. Therefore,

$$\begin{aligned} I(\phi) &= \sum_{i=1}^n a_i \mu^*(A_i) = \sum_{i=j}^m \sum_{i=1}^n a_i \mu^*(A_i \cap B_j) \\ &= \sum_{j=1}^m \sum_{i=1}^n b_j \mu^*(A_i \cap B_j) = \sum_{j=1}^m b_j \mu^*(B_j). \end{aligned}$$

Now, consider the general case. By Problem 12.14, there exist pairwise disjoint measurable sets $C_1, \ldots, C_k$ such that each C_i is included in some B_j and $B_j = \bigcup\{C_i: C_i \subseteq B_j\}$. For each i and j let $\delta_i^j = 1$ if $C_i \subseteq B_j$ and $\delta_i^j = 0$ if $C_i \nsubseteq B_j$.

Clearly, $\chi_{B_j} = \sum_{i=1}^{k} \delta_i^j \chi_{C_i}$ and $\mu(B_j) = \sum_{i=1}^{k} \delta_i^j \mu^*(C_i)$. Therefore,

$$\phi = \sum_{j=1}^{m} b_j \chi_{B_j} = \sum_{j=1}^{m} b_j \Big[\sum_{i=1}^{k} \delta_i^j \chi_{C_i} \Big] = \sum_{i=1}^{k} \Big[\sum_{j=1}^{m} b_j \delta_i^j \Big] \chi_{C_i}.$$

So, by the preceding case, we have

$$I(\phi) = \sum_{i=1}^{k} \Big[\sum_{j=1}^{m} b_j \delta_i^j \Big] \mu^*(C_i) = \sum_{j=1}^{m} b_j \Big[\sum_{i=1}^{k} \delta_i^j \mu^*(C_i) \Big] = \sum_{j=1}^{m} b_j \mu^*(B_j).$$

Problem 17.5. *Show that $|I(\phi)| \le I(|\phi|)$ holds for every step function ϕ.*

Solution. From $-|\phi| \le \phi \le |\phi|$ and the monotonicity of the integral (Theorem 17.3), it follows that

$$-I(|\phi|) = I(-|\phi|) \le I(\phi) \le I(|\phi|),$$

and so $|I(\phi)| \le I(|\phi|)$ holds.

Problem 17.6. *Let ϕ be a step function such that $I(|\phi|) = 0$. Show that $\phi = 0$ a.e. holds.*

Solution. Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ be the standard representation of ϕ. Then, note that $|\phi| = \sum_{i=1}^{n} |a_i| \chi_{A_i}$ is a representation of $|\phi|$, and therefore

$$0 = I(|\phi|) = \sum_{i=1}^{n} |a_i| \mu^*(A_i).$$

Since $|a_i| > 0$ holds for each $1 \le i \le n$, it follows that $\mu^*(A_i) = 0$ for each $1 \le i \le n$, and so $\phi = 0$ a.e. holds.

Problem 17.7. *Let ϕ be a step function. Let $A = \{x \in X\colon \phi(x) \ne 0\}$ and $M = \max\{|\phi(x)|\colon x \in X\}$. Show that $|I(\phi)| \le M\mu^*(A)$.*

Solution. Apply the monotonicity of the integral (Theorem 17.3) to the inequality $-M\chi_A \le \phi \le M\chi_A$.

Problem 17.8. *Let $\{\phi_n\}$ be a sequence of step functions. Show that if ϕ is a step function and $\phi_n \downarrow \phi$ a.e. holds, then $I(\phi_n) \downarrow I(\phi)$ also holds.*

Solution. If $\phi_n \downarrow \phi$ a.e. holds, then $\phi_n - \phi \downarrow 0$ a.e. likewise holds. Thus, by the order continuity of the integral (Theorem 17.4), $I(\phi_n) - I(\phi) = I(\phi_n - \phi) \downarrow 0$ so that $I(\phi_n) \downarrow I(\phi)$.

Problem 17.9. *Let $\{\phi_n\}$ be a sequence of step functions and ϕ a simple function such that $0 \le \phi_n \uparrow \phi$ a.e. holds. Show that if $\lim I(\phi_n) < \infty$, then ϕ is a step function.*

Solution. Assume $\phi(x) \ge 0$ for each x and let $\phi = \sum_{i=1}^{k} a_i \chi_{A_i}$ be the standard representation of ϕ. Now, let i be fixed. Then, for each n the function $\psi_n = \phi_n \wedge a_i \chi_{A_i}$ is a step function, $\psi_n \le \phi_n$ holds, and $\psi_n \uparrow_n \phi \wedge a_i \chi_{A_i} = a_i \chi_{A_i}$ a.e. By Theorem 17.6, we see that

$$0 \le a_i \mu^*(A_i) = \lim_{n\to\infty} I(\psi_n) \le \lim_{n\to\infty} I(\phi_n) < \infty,$$

and so $\mu^*(A_i) < \infty$ holds for each $1 \le i \le k$. That is, ϕ is a step function.

Problem 17.10. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f: X \to \mathbf{R}$ be a function. Show that f is a measurable function if and only if there exists a sequence $\{\phi_n\}$ of simple functions such that $\lim \phi_n(x) = f(x)$ holds for all $x \in X$.*

Solution. Assume f to be measurable. Then both f^+ and f^- are measurable functions. By Theorem 17.7 there exist two sequences of simple functions $\{s_n\}$ and $\{t_n\}$ with $0 \le s_n(x) \uparrow f^+(x)$ and $0 \le t_n(x) \uparrow f^-(x)$ for each $x \in X$. Now, note that $\phi_n = s_n - t_n$ satisfies $\phi_n(x) \longrightarrow f(x)$ for all x.

For the converse, note that (by Theorem 16.6) the pointwise limit of a sequence of measurable functions is always a measurable function.

Problem 17.11. *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, and let $f: X \to \mathbf{R}$ be a measurable function such that $f(x) \ge 0$ for all $x \in X$. Show that there exists a sequence $\{\phi_n\}$ of step functions such that $0 \le \phi_n \uparrow f(x)$ holds for all $x \in X$.*

Solution. By Theorem 17.7 there exists a sequence $\{\psi_n\}$ of simple functions satisfying $0 \le \psi_n(x) \uparrow f(x)$ for all $x \in X$. Now, pick a sequence $\{E_n\}$ of measurable sets with $\mu^*(E_n) < \infty$ for each n, and $E_n \uparrow X$. Let $\phi_n = \psi_n \wedge \chi_{E_n}$. Then, $\{\phi_n\}$ is a sequence of step functions satisfying $0 \le \phi_n(x) \uparrow f(x)$ for each $x \in X$.

Problem 17.12. *Give a proof of the order continuity of the integral, i.e., $\phi_n \downarrow 0$ a.e. implies $I(\phi_n) \downarrow 0$, based on Egorov's Theorem 16.7.*

Solution. Assume that $\{\phi_n\}$ is a sequence of step functions of some measure space $(X, \mathcal{S}, \mu)$ satisfying $\phi_n \downarrow 0$ a.e. Without loss of generality, we can suppose that $\phi_n(x) \downarrow 0$ for each $x \in X$. Let $E = \{x \in X\colon \phi_1(x) > 0\}$ and note that $\mu^*(E) < \infty$. Also, let $M = \max\{\phi_1(x)\colon x \in X\}$.

Now, let $\epsilon > 0$. By Egorov's Theorem 16.7 there exists a measurable set $F \subseteq E$ such that $\mu^*(F) < \epsilon$ and $\{\phi_n\}$ converges uniformly to zero on $E \setminus F$. So, there exists some k such that $0 \leq \phi_n(x) < \epsilon$ for all $x \in E \setminus F$ and all $n \geq k$. Thus, for $n \geq k$, we have

$$0 \leq \phi_n \leq \epsilon\chi_{E\setminus F} + M\chi_F \leq \epsilon\chi_E + M\chi_F,$$

and consequently, by the monotonicity of the integral

$$0 \leq I(\phi_n) \leq \epsilon\mu^*(E) + M\mu^*(F) < \left[\mu^*(E) + M\right]\epsilon$$

for all $n \geq k$. This shows that $I(\phi_n) \downarrow 0$.

Problem 17.13. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f\colon X \to [0, \infty)$ be a function. Show that f is measurable if and only if there exist non-negative constants $c_1, c_2, \ldots$ and measurable sets $E_1, E_2, \ldots$ such that*

$$f(x) = \sum_{n=1}^{\infty} c_n \chi_{E_n}(x)$$

holds for each $x \in X$.

Solution. Consider a measure space $(X, \mathcal{S}, \mu)$ and a non-negative real-valued function $f\colon X \to [0, \infty)$. Assume first that there exist non-negative constants $c_1, c_2, \ldots$ and measurable sets $E_1, E_2, \ldots$ such that $f(x) = \sum_{n=1}^{\infty} c_n\chi_{E_n}(x)$ holds for each $x \in X$. If we let $\phi_n = \sum_{i=1}^{n} c_i\chi_{E_i}$, then ϕ_n is a measurable function (in fact, it is a simple function) and $\phi_n(x) \longrightarrow f(x)$ holds for each $x \in X$. Now, by Theorem 16.6(1), the function f is necessarily a measurable function.

For the converse, assume that f is a measurable function. By Theorem 17.7 there exists a sequence of simple functions $\{\phi_n\}$ such that $0 \leq \phi_n(x) \uparrow f(x)$ for each $x \in X$. If we let $\phi_0 = 0$, then $f(x) = \sum_{n=1}^{\infty}\left[\phi_n(x) - \phi_{n-1}(x)\right]$ holds for each $x \in X$. For each n write $\phi_n - \phi_{n-1} = \sum_{i=1}^{k_n} c_i^n\chi_{E_i^n}$ with $c_i^n \geq 0$ for each i and n. Thus,

$$f(x) = \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} c_i^n\chi_{E_i^n},$$

and by rearranging the terms of the preceding series in a single series, our conclusion follows.

Problem 17.14. *Let (X, S, μ) be a finite measure space satisfying $\mu^*(X) = 1$, and let $E_1, E_2, \ldots, E_{10}$ be ten measurable sets such that $\mu^*(E_i) = \frac{1}{3}$ holds for each i. Show that four of these sets have an intersection of positive measure. Is the conclusion true for nine measurable sets instead of ten?*

Solution. Consider the step function $\phi = \sum_{i=1}^{10} \chi_{E_i}$. Clearly, the function ϕ assumes only integer values and

$$\phi(x) = \text{the cardinality of the set } \big\{i \in \{1, \ldots, 10\}\colon x \in E_i\big\}.$$

If $\phi(x) \leq 3 = 3\chi_X(x)$ for almost all x, then

$$3 < \tfrac{10}{3} = \sum_{i=1}^{10} \mu^*(E_i) = \sum_{i=1}^{10} I\big(\chi_{E_i}\big) = I(\phi) \leq I\big(3\chi_X\big) = 3$$

a contradiction. Hence, the measurable set

$$A = \big\{x \in X\colon \phi(x) \geq 4\big\}$$

must have positive measure.

Next, let $A_1, A_2, \ldots, A_k$ denote the collection of all (nonempty) intersections of the sets E_i taken four at a time; clearly, $k \leq \binom{10}{4} = 210$. Now, an easy argument guarantees that $A \subseteq \bigcup_{j=1}^{k} A_j$, and from this it easily follows that at least one of the A_j must have positive measure.

For nine sets the conclusion is false. For a counterexample take $X = [0, 1]$, $\mu = \lambda$, $E_1 = E_2 = E_3 = \big(0, \frac{1}{3}\big)$, $E_4 = E_5 = E_6 = \big(\frac{1}{3}, \frac{2}{3}\big)$, and $E_7 = E_8 = E_9 = \big(\frac{2}{3}, 1\big)$.

Problem 17.15. *If $f\colon X \to [0, 1]$ is a measurable function, then show that either $f = \chi_A$ a.e. for some measurable set A or else (exclusively) there exists a constant $0 < c < \frac{1}{2}$ such that*

$$\mu^*(\{x \in X\colon c < f(x) < 1 - c\}) > 0.$$

Solution. For each n let $A_n = \big\{x \in X\colon \frac{1}{2n} < f(x) < 1 - \frac{1}{2n}\big\}$. If $\mu^*(A_n) > 0$ for some n, then the constant $c = \frac{1}{2n}$ satisfies $\mu^*\big(\{x \in X\colon c < f(x) < 1 - c\}\big) > 0$.

Now, assume that $\mu^*(A_n) = 0$ for each n. Then from

$$A_n \uparrow \{x \in X: 0 < f(x) < 1\},$$

we see that $\mu^*\big(\{x \in X: 0 < f(x) < 1\}\big) = 0$. This easily implies that $f = \chi_A$ a.e. for the measurable set $A = f^{-1}(\{1\})$.

Problem 17.16. *Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $\phi: X \to \mathbb{R}$ be a simple function having the standard representation $\phi = \sum_{i=1}^n a_i \chi_{A_i}$. If $\phi \geq 0$ a.e., then the sum $\sum_{i=1}^n a_i \mu^*(A_i)$ makes sense as an extended real number (it may be infinite). Call this extended real number the* **Lebesgue integral** *of ϕ, and write $I(\phi) = \sum_{i=1}^n a_i \mu^*(A_i)$.*

a. *If ϕ and ψ are simple functions such that $\phi \geq 0$ a.e., then $\psi \geq 0$ a.e., then show that $I(\phi + \psi) = I(\phi) + I(\psi)$.*
b. *If ϕ and ψ are simple functions such that $0 \leq \phi \leq \psi$ a.e., then show that $I(\phi) \leq I(\psi)$.*
c. *Show that if $\{\phi_n\}$ and $\{\psi_n\}$ are two sequences of simple functions and $f: X \to \mathbb{R}^*$ such that $0 \leq \phi_n \uparrow f$ a.e. and $0 \leq \psi_n \uparrow f$ a.e., then $\lim I(\phi_n) = \lim I(\psi_n)$ holds (with the limits possibly being infinite).*
d. *Assume that $\{\phi_n\}$ is a sequence of simple functions such that $0 \leq \phi_n \uparrow \chi_A$ a.e. holds. Show that $\lim I(\phi_n) = \mu^*(A)$.*
e. *Give an example of a sequence $\{\phi_n\}$ of simple functions on some measure space such that $\phi_n \downarrow 0$ (everywhere) and $\lim I(\phi_n) \neq 0$.*

Solution. Clearly, a simple function ϕ is a step function if and only if $I(\phi) < \infty$.

(a) Note that $\phi + \psi$ is a step function if and only if both ϕ and ψ are step functions. In this case, the equality $I(\phi + \psi) = I(\phi) + I(\psi)$ follows from Theorem 17.2. On the other hand, if $\phi + \psi$ is not a step function, then either ϕ or ψ fails to be a step function and hence, in this case, $I(\phi + \psi) = I(\phi) + I(\psi) = \infty$ holds.

(b) If $I(\psi) = \infty$, then $I(\phi) \leq I(\psi)$ holds trivially. On the other hand, if $I(\psi) < \infty$, then ψ is a step function. It follows (from Problem 17.2) that ϕ is a step function, and the desired inequality follows from Theorem 17.3.

(c) If both $\{\phi_n\}$ and $\{\psi_n\}$ are sequences of step functions, then the conclusion follows from Theorem 17.5. Thus, we only need to consider the case when $\{\phi_n\}$ is a sequence of step functions and $I(\psi_k) = \infty$ holds for some k.

In view of $\phi_n \wedge \psi_k \uparrow_n f \wedge \psi_k = \psi_k$ a.e., it follows from Problem 17.9 that $\lim_{n\to\infty} I(\phi_n \wedge \psi_k) = \infty$. From $\phi_n \wedge \psi_k \leq \phi_n$, we obtain that $\lim I(\phi_n) = \infty$. Hence, $\lim I(\psi_n) = \lim I(\phi_n) = \infty$ holds in this case.

(d) We can suppose $0 \leq \phi_n(x) \uparrow \chi_A(x)$ holds for each x. If for each n we let $A_n = \{x \in X: \phi_n(x) > 0\}$, then each A_n is measurable and $A_n \uparrow A$ holds. Since

$\chi_{A_n} \uparrow \chi_A$, part (c) coupled with Theorem 15.4 gives

$$\lim_{n\to\infty} I(\phi_n) = \lim_{n\to\infty} I(\chi_{A_n}) = \lim_{n\to\infty} \mu^*(A_n) = \mu^*(A).$$

(e) Consider $\mathbf{R}$ with the Lebesgue measure, and let $\phi_n = \chi_{(n,\infty)}$.

Problem 17.17. *Let (X, Σ, μ) be a measure space with Σ being a σ-algebra. Let us say that a function $f: X \to \mathbf{R}$ is Σ-measurable if $f^{-1}(A) \in \Sigma$ for each open subset A of $\mathbf{R}$. Also, let $\mathcal{M}_\Sigma$ denote the collection of all Σ-measurable functions. Establish the following:*

a. *$\mathcal{M}_\Sigma$ is a function space and an algebra of functions.*
b. *$\mathcal{M}_\Sigma$ is closed under sequential pointwise limits.*
c. *If μ is σ-finite and $f: X \to \mathbf{R}$ is a measurable function, then there exists a Σ-measurable function $g: X \to \mathbf{R}$ such that $f = g$ a.e.*

Solution. (a) In order to show that $\mathcal{M}_\Sigma$ is closed under addition and multiplication, we need the following properties among Σ-measurable functions f and g: The sets

1. $\{x \in X: f(x) > g(x)\}$,
2. $\{x \in X: f(x) \geq g(x)\}$, and
3. $\{x \in X: f(x) = g(x)\}$

all belong to Σ. To see (1), let $r_1, r_2, \ldots$ be an enumeration of the rational numbers of $\mathbf{R}$, and note that

$$\{x \in X: f(x) > g(x)\} = \bigcup_{n=1}^{\infty} \Big[\{x \in X: f(x) > r_n\} \cap \{x \in X: g(x) < r_n\}\Big],$$

which belongs to Σ, since it is a countable union of sets from the σ-algebra Σ. For (2), note that $\{x \in X: f(x) \geq g(x)\} = \{x \in X: g(x) > f(x)\}^c$, which belongs to Σ by (1). Finally, for (3), observe that

$$\{x \in X: f(x) = g(x)\} = \{x \in X: f(x) \geq g(x)\} \cap \{x \in X: g(x) \geq f(x)\},$$

which belongs to Σ by (2).

To complete the proof of part (a), we shall establish that for Σ-measurable functions f and g, the following statements hold:

i. $f + g$ is a Σ-measurable function.
ii. fg is a Σ-measurable function.

iii. $|f|$, f^+, and f^- are Σ-measurable functions.
iv. $f \vee g$ and $f \wedge g$ are Σ-measurable functions.

The proofs of these claims are given below.

(i) Note first that if c is a constant number, then $c - g$ is a Σ-measurable function. [Reason: If $a \in \mathbf{R}$, then $\{x \in X\colon\ c - g(x) \geq a\} = \{x \in X\colon\ g(x) \leq c - a\} \in \Sigma$.] Now, if $a \in \mathbf{R}$, then the set

$$(f+g)^{-1}\big([a,\infty)\big) = \big\{x \in X\colon\ f(x) + g(x) \geq a\big\} = \big\{x \in X\colon\ f(x) \geq a - g(x)\big\}$$

belongs to Σ by the preceding observation and (2). This implies (how?) that $f+g$ is a Σ-measurable function.

(ii) Note first that f^2 is a Σ- measurable function. To see this, let $a \in \mathbf{R}$. Then $\big\{x \in X\colon\ f^2(x) \leq a\big\} = \emptyset$ if $a < 0$ and $\big\{x \in X\colon\ f^2(x) \leq a\big\} = f^{-1}\big([-\sqrt{a}, \sqrt{a}\,]\big)$ if $a \geq 0$. This implies that f^2 is a Σ-measurable function. Also, if c is a constant, then cf is measurable. [Reason: If $A = \{x \in X\colon\ cf(x) \geq a\}$, then $A = \big\{x \in X\colon\ f(x) \geq a/c\big\}$ for $c > 0$ and $A = \big\{x \in X\colon\ f(x) \leq a/c\big\}$ for $c < 0$.] The result now follows from the preceding observations combined with (i) and the relation

$$fg = \tfrac{1}{2}\big[(f+g)^2 - f^2 - g^2\big].$$

(iii) The Σ-measurability of $|f|$ follows from the relation

$$\big\{x \in X\colon\ |f(x)| \leq a\big\} = \emptyset \quad \text{if } a < 0,$$

and

$$\big\{x \in X\colon\ |f(x)| \leq a\big\} = \{x \in X\colon\ f(x) \leq a\} \cap \{x \in X\colon\ f(x) \geq -a\} \quad \text{if } a \geq 0.$$

For the Σ-measurability of f^+ and f^- use the identities

$$f^+ = \tfrac{1}{2}\big(|f| + f\big) \quad \text{and} \quad f^- = \tfrac{1}{2}\big(|f| - f\big).$$

(iv) The identities

$$f \vee g = \tfrac{1}{2}\big(f + g + |f - g|\big) \quad \text{and} \quad f \wedge g = \tfrac{1}{2}\big(f + g - |f - g|\big)$$

show that $f \vee g$ and $f \wedge g$ are Σ-measurable functions.

(b) Assume that $\{f_n\}$ is a sequence of Σ-measurable functions such that $f_n(x) \to f(x)$ holds for each $x \in X$. Observe that the equality

$$f^{-1}\big((a,\infty)\big) = \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} f_i^{-1}\big((a+\tfrac{1}{n},\infty)\big)$$

and the Σ-measurability of each f_i show that $f^{-1}\big((a,\infty)\big)$ belongs to Σ. This implies that f is a Σ-measurable function.

(c) We can assume $f(x) \geq 0$ for each $x \in X$ (otherwise, we apply the arguments below to f^+ and f^- separately). Assume first that $f = \chi_A$ for some $A \in \Sigma$. Since μ is σ-finite, it follows from Theorem 15.11 that there exists a μ-null set C such that $B = A \cup C \in \Sigma$. So, if $g = \chi_B$, then g is Σ-measurable and $f = g$ μ-a.e. It follows that if ϕ is a μ-simple function, then there exists a Σ-simple function ψ such that $\psi = \phi$ μ-a.e.

Now, by Theorem 17.7, there exists a sequence $\{\phi_n\}$ of simple functions such that $\phi_n(x) \uparrow f(x)$ for each $x \in X$. Replacing each ϕ_n by a Σ-simple function ψ_n (as above) we have $\psi_n(x) \uparrow f(x)$ for μ-almost all x. So, there exists a μ-measurable set E such that $\psi_n(x) \uparrow f(x)$ for each $x \notin E$. Now, use Theorem 15.11 to select a set $F \in \Sigma$ with $E \subseteq F$ and $\mu^*(F) = 0$. Clearly, $\psi_n(x)\chi_{F^c}(x) \uparrow f(x)\chi_{F^c}(x) = g(x)$ for each $x \in X$. By part (b), g is Σ-measurable and satisfies $g = f$ μ-a.e.

18. THE LEBESGUE MEASURE

Problem 18.1. *Let $I = \prod_{i=1}^n I_i$ be an interval of $\mathbb{R}^n$. Show that I is Lebesgue measurable and that $\lambda(I) = \prod_{i=1}^n |I_i|$, where $|I_i|$ denotes the length of the interval I_i.*

Solution. The verification of the formula can be done by cases as in Problem 15.3. To show this, we establish the formula for two cases, and leave the rest for the reader.

The first case is when $I_i = [a_i, b_i]$, where $-\infty < a_i < b_i < \infty$ holds for each $1 \leq i \leq n$. Then, $\Pi_{i=1}^n [a_i, b_i + \frac{1}{k}) \downarrow_k \Pi_{i=1}^n I_i = I$. Thus, from Theorem 15.4, it follows that

$$\begin{aligned}\lambda(I) &= \lim_{k\to\infty} \lambda\Big(\prod_{i=1}^n [a_i, b_i + \tfrac{1}{k})\Big) = \lim_{k\to\infty} \prod_{i=1}^n (b_i - a_i + \tfrac{1}{k})\\ &= \prod_{i=1}^n (b_i - a_i) = \prod_{i=1}^n |I_i|.\end{aligned}$$

The second case is when $I = [a,\infty) \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. Then, note that $[a, a+k] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \uparrow_k I$. Taking into account the preceding

case, it follows from Theorem 15.4 that

$$\lambda(I) = \lim_{k\to\infty} \lambda\big([a, a+k] \times [a_2, b_2] \times \cdots \times [a_n, b_n]\big)$$

$$= \lim_{k\to\infty} k \cdot (b_2 - a_2) \cdots (b_n - a_n) = \infty = \prod_{i=1}^{\infty} |I_i|.$$

Problem 18.2. *Let $\mathcal{O}$ be an open subset of $\mathbf{R}$. Show that there exists an at-most countable collection $\{I_\alpha\colon \alpha \in A\}$ of pairwise disjoint open intervals such that $\mathcal{O} = \bigcup_{\alpha\in A} I_\alpha$. Also, show that $\lambda(\mathcal{O}) = \sum_{\alpha\in A} |I_\alpha|$.*

Solution. Let $\mathcal{O}$ be an open subset of $\mathbf{R}$. By part (g) of Problem 6.11, we know that there exists an at-most countable collection $\{I_\alpha\colon \alpha \in A\}$ of pairwise disjoint open intervals such that $\mathcal{O} = \bigcup_{\alpha\in A} I_\alpha$.

Now, using the fact that the length of each I_α coincides with its Lebesgue measure (Problem 15.3), we see that

$$\lambda(\mathcal{O}) = \lambda\Big(\bigcup_{\alpha\in A} I_\alpha\Big) = \sum_{\alpha\in A} \lambda(I_\alpha) = \sum_{\alpha\in A} |I_\alpha|.$$

Problem 18.3. *Show that the Borel sets of $\mathbf{R}^n$ are precisely the members of the σ-algebra generated by the compact sets.*

Solution. Let $\mathcal{C}$ denote the σ-algebra generated by the compact sets. Since every compact set is closed (which is the complement of an open set), it follows that $\mathcal{C} \subseteq \mathcal{B}$. On the other hand, if C is a closed set and $C_n = \{x \in C\colon d(0, x) \le n\}$, then $\{C_n\}$ is a sequence of compact sets satisfying $C_n \uparrow C$. This implies that $\mathcal{C}$ contains all the closed sets (and hence, all the open sets). Thus, $\mathcal{B} \subseteq \mathcal{C}$ also holds, and so $\mathcal{B} = \mathcal{C}$.

Problem 18.4. *Show that a subset E of $\mathbf{R}^n$ is Lebesgue measurable if and only if for each $\epsilon > 0$ there exists a closed subset F of $\mathbf{R}^n$ such that $F \subseteq E$ and $\lambda(E \setminus F) < \epsilon$.*

Solution. Assume that E is Lebesgue measurable and let $\varepsilon > 0$. Since E^c is also Lebesgue measurable, there exists an open set V such that $E^c \subseteq V$ and $\lambda(V \setminus E^c) = \lambda(E \cap V) < \varepsilon$. Then, the closed set $C = V^c$ satisfies $C \subseteq E$ and $\lambda(E \setminus C) = \lambda(E \cap V) < \varepsilon$. For the converse, either reverse the preceding arguments and use Theorem 18.2, or else use Problem 14.8.

Problem 18.5. *Show that if a subset E of $[0, 1]$ satisfies $\lambda(E) = 1$, then E is dense in $[0, 1]$.*

Solution. Let I be a (nonempty) subinterval of $[0, 1]$. If $I \cap E = \emptyset$, then we have $\lambda(E) + \lambda(I) = \lambda(E \cup I) \leq \lambda([0, 1]) = 1$, and hence, in this case, $\lambda(E) \leq 1 - \lambda(I) < 1$ holds, which is a contradiction. Thus, $I \cap E \neq \emptyset$ holds for each subinterval I of $[0, 1]$, and so the set E is dense in $[0, 1]$.

Problem 18.6. *If $E \subseteq \mathbf{R}^n$ satisfies $\lambda(E) = 0$, then show that $E^o = \emptyset$.*

Solution. If V is a nonempty open set with $V \subset E$, then note that $0 < \lambda(V) \leq \lambda(E)$ holds. Therefore, the open set E^o must be empty.

Problem 18.7. *Show that if E is a Lebesgue measurable subset of $\mathbf{R}^n$, then there exist an F_σ-set A and a G_δ-set B such that $A \subseteq E \subseteq B$ and $\lambda(B \setminus A) = 0$.*

Solution. By Problem 18.4, for each k there exists a closed set C_k with $C_k \subseteq E$ and $\lambda(E \setminus C_k) < \frac{1}{k}$. Similarly, by Theorem 18.2, for every k there exists an open set V_k with $E \subseteq V_k$ and $\lambda(V_k \setminus E) < \frac{1}{k}$. Put $A = \bigcup_{k=1}^{\infty} C_k$ (an F_σ-set) and $B = \bigcap_{k=1}^{\infty} V_k$ (a G_δ-set). Clearly, $A \subseteq E \subseteq B$ holds, and in view of

$$\begin{aligned} \lambda(B \setminus A) &\leq \lambda(V_k \setminus C_k) = \lambda\big((V_k \setminus E) \cup (E \setminus C_k)\big) \\ &\leq \lambda(V_k \setminus E) + \lambda(E \setminus C_k) < \tfrac{2}{k} \end{aligned}$$

for each k, we see that $\lambda(B \setminus A) = 0$.

Problem 18.8. *Let $\{E_n\}$ be a sequence of nonempty (Lebesgue) measurable subsets of $[0, 1]$ satisfying $\lim \lambda(E_n) = 1$.*

a. *Show that for each $0 < \epsilon < 1$ there exists a subsequence $\{E_{k_n}\}$ of $\{E_n\}$ such that $\lambda(\bigcap_{n=1}^{\infty} E_{k_n}) > \epsilon$.*
b. *Show that $\bigcap_{k=n}^{\infty} E_k = \emptyset$ is possible for each $n = 1, 2, \ldots$.*

Solution. Let $\{E_n\}$ be a sequence of nonempty Lebesgue measurable subsets of $[0, 1]$ satisfying $\lim \lambda(E_n) = 1$.

(a) Fix $0 < \varepsilon < 1$. From $\lim \lambda(E_n) = 1$, we see that there exists a subsequence $\{E_{k_n}\}$ of $\{E_n\}$ satisfying $\lambda(E_{k_n}) > 1 - \frac{1-\varepsilon}{2^n}$. Now, consider the measurable sets $E = \bigcap_{n=1}^{\infty} E_{k_n}$ and $F = [0, 1] \setminus E$. Then, we have

$$\begin{aligned} \lambda(F) &= \lambda([0, 1] \setminus E) = \lambda\Big(\bigcup_{n=1}^{\infty}([0, 1] \setminus E_{k_n})\Big) \\ &= \sum_{n=1}^{\infty} \lambda([0, 1] \setminus E_{k_n}) = \sum_{n=1}^{\infty}[1 - \lambda(E_{k_n})] < \sum_{n=1}^{\infty} \tfrac{1-\varepsilon}{2^n} = 1 - \varepsilon. \end{aligned}$$

Hence, $\lambda(E) = 1 - \lambda(F) > 1 - (1 - \varepsilon) = \varepsilon$.

(b) Let $A_k^n = [0,1] \setminus \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $1 \le k \le n$; $n \ge 2$. Clearly, $\lambda(A_k^n) = 1 - \frac{1}{n}$ holds for each $1 \le k \le n$ and $\bigcap_{k=1}^{n} A_k^n = \emptyset$ holds for each $n \ge 2$. Let E_n denote the sequence

$$A_1^2,\ A_2^2,\ A_1^3,\ A_2^3,\ A_3^3,\ \ldots,\ A_1^n,\ A_2^n,\ \ldots,\ A_n^n,\ A_1^{n+1},\ \ldots.$$

Now, note that $\lambda(E_n) \longrightarrow 1$ and $\bigcap_{k=n}^{\infty} E_k = \emptyset$ holds for each $n \ge 1$.

Problem 18.9. *Assume that a function $f: I \to \mathbb{R}$ defined on a subinterval of $\mathbb{R}$ satisfies a Lipschitz condition. That is, assume that there exists a constant $C > 0$ such that $|f(x) - f(y)| \le C|x - y|$ holds for all $x, y \in I$. Show that f carries (Lebesgue) null sets to null sets.*

In particular, if a function $f: I \to \mathbb{R}$ defined on a subinterval of $\mathbb{R}$ has a continuous derivative, then show that f carries null sets to null sets.

Solution. Assume that a function $f: I \longrightarrow \mathbb{R}$ satisfies the condition of the problem. Clearly, f is a (uniformly) continuous function. In particular, note that if J is a subinterval of I, then $f(J)$ is also a subinterval of $\mathbb{R}$ (see part (g) of Problem 6.11), and our condition implies (how?) that the length of $f(J)$ is less than or equal to C times the length of J, i.e., $\lambda^*\big(f(J)\big) \le C\lambda^*(J)$ holds.

Now, let A be a null subset of I and let $\varepsilon > 0$. Pick a sequence $\{[a_n, b_n)\}$ of half-open intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda^*\big([a_n, b_n)\big) = \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$

Hence, $f(A) \subseteq f\big(\bigcup_{n=1}^{\infty} [a_n, b_n) \cap I\big) = \bigcup_{n=1}^{\infty} f\big([a_n, b_n) \cap I\big)$, and so by the preceding

$$\begin{aligned}
\lambda^*\big(f(A)\big) &\le \lambda^*\Big(\bigcup_{n=1}^{\infty} f\big([a_n, b_n) \cap I\big)\Big) \\
&\le \sum_{n=1}^{\infty} \lambda^*\big(f\big([a_n, b_n) \cap I\big)\big) \le \sum_{n=1}^{\infty} C(b_n - a_n) < C\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we see that $\lambda^*\big(f(A)\big) = 0$ holds, as desired.

For the second part notice that if $[a, b]$ is a closed subinterval of I, then there exists some constant $M > 0$ satisfying $|f'(t)| \le M$ for all $t \in [a, b]$. Now, if $x, y \in [a, b]$, then there exists (by the Mean Value Theorem) some z between x and y satisfying $f(x) - f(y) = f'(z)(x - y)$. This implies $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in [a, b]$. So, by the first part, f carries null sets of $[a, b]$ to null sets.

Now, fix a sequence $\{[a_n, b_n]\}$ of closed subintervals of I such that $I = \bigcup_{n=1}^{\infty} [a_n, b_n]$ and let A be a null subset of I. Then $A \cap [a_n, b_n]$ is a null subset of $[a_n, b_n]$, and so $f(A \cap [a_n, b_n])$ is a null subset of $\mathbf{R}$. Now, notice that the identity

$$f(A) = f\Big(\bigcup_{n=1}^{\infty} A \cap [a_n, b_n]\Big) = \bigcup_{n=1}^{\infty} f(A \cap [a_n, b_n])$$

guarantees that $f(A)$ is a null subset of $\mathbf{R}$.

Problem 18.10. *Show that the Lebesgue measure of a triangle in $\mathbf{R}^2$ equals its area. Also, determine the Lebesgue measure of a disk in $\mathbf{R}^2$.*

Solution. Start by observing that every line segment has Lebesgue measure zero (why?). Thus, the Lebesgue measure of a triangle is the same with or without some of its edges. Also, every triangle is Lebesgue measurable (since without its edges it is an open set). Since λ is translation invariant, we can assume that all triangles have one of their vertices at zero. Let T be such a triangle, and let $A(T)$ denote its area. Following the graphs in Figure 3.1 (from left to right) we see that:

$$\begin{aligned} 2\lambda(T) &= \lambda(T) + \lambda(-T) = \lambda(T) + \lambda(T_2) = \lambda(T) + \lambda(T_1) \\ &= \lambda(T_1 \cup T) = \lambda(P) = \lambda(Q) = A(P) = 2A(T). \end{aligned}$$

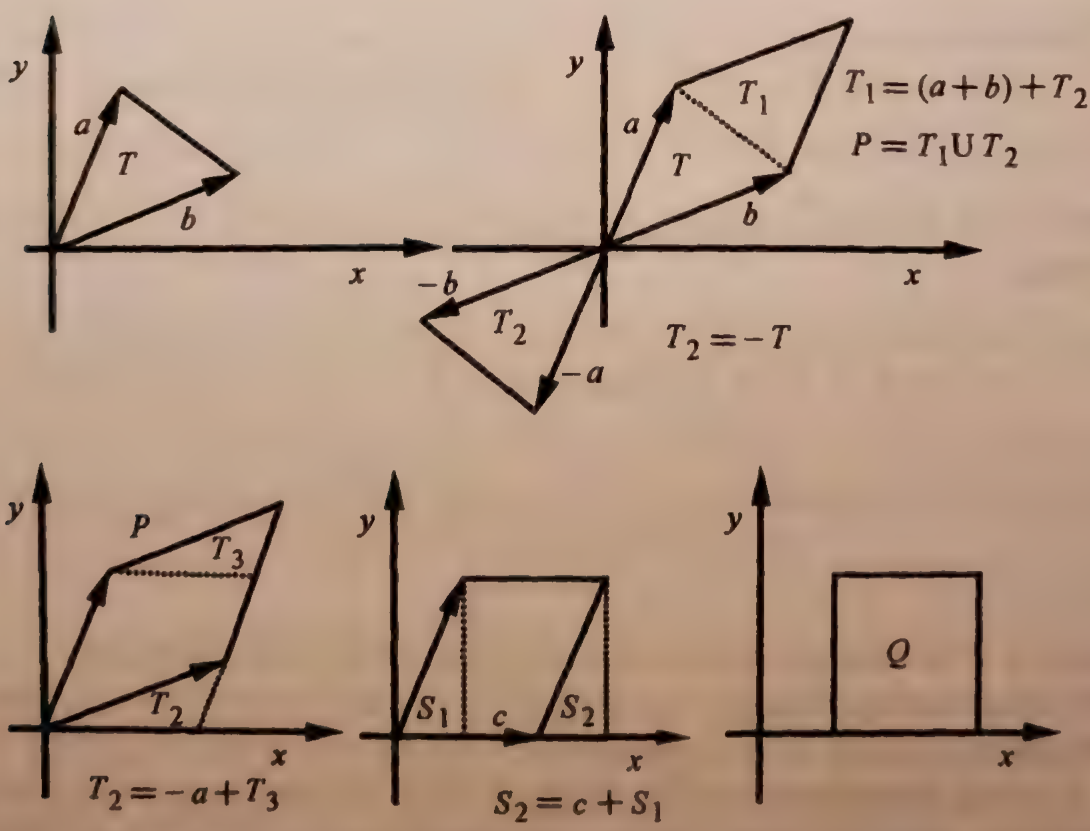

FIGURE 3.1. The Lebesgue Measure of a Triangle

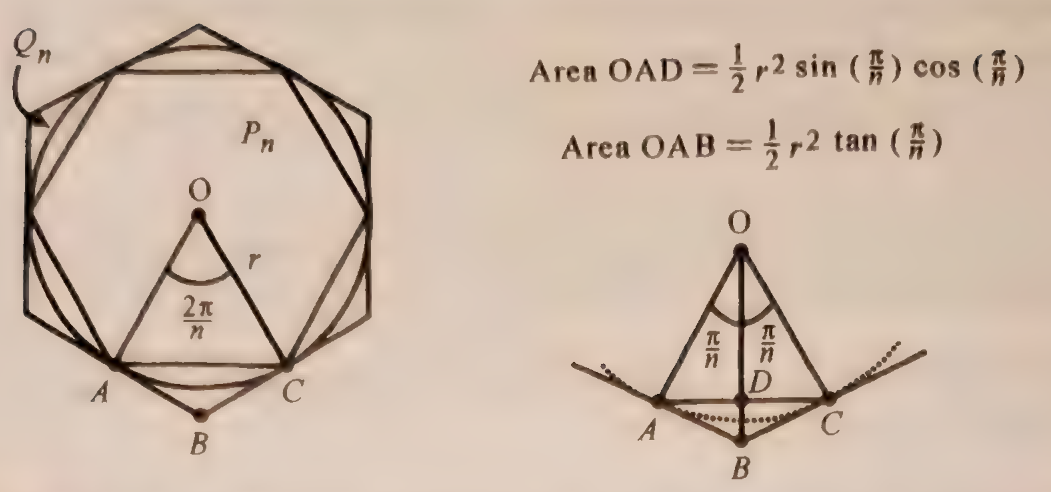

FIGURE 3.2. The Computation of the Lebesgue Measure of a Disk

That is, $\lambda(T) = A(T)$. In particular, this implies that the Lebesgue measure of any polygon equals its area.

Now, let D be a closed disk of radius r; see Figure 3.2. To compute its Lebesgue measure, we use the Eudoxus–Archimedes *Method of Exhaustion*. For each n, let P_n and Q_n be the inscribed and circumscribed regular n-polygons, respectively. Clearly, $P_n \subseteq D \subseteq Q_n$ holds. Now, note that

$$\lambda(P_n) = \pi r^2 \left[\frac{\sin(\frac{\pi}{n})}{\frac{\pi}{n}}\right] \cos(\tfrac{\pi}{n}) \le \lambda(D) \le \lambda(Q_n) = \pi r^2 \left[\frac{\tan(\frac{\pi}{n})}{\frac{\pi}{n}}\right],$$

and so, by letting $n \longrightarrow \infty$, we see that

$$\lambda(D) = \pi r^2 = A(D).$$

Problem 18.11. *If μ is a translation invariant Borel measure on $\mathbf{R}^n$, then show that there exists some $c \ge 0$ such that $\mu^*(A) = c\lambda^*(A)$ for all subset A of $\mathbf{R}^n$.*

Solution. By Theorem 18.8, $\mu = c\lambda$ holds on $\mathcal{B}$ for some constant $c \ge 0$. Now, by Theorem 14.10, $(c\lambda)^* = c\lambda^*$ holds, and consequently $\mu^*(A) = (c\lambda)^*(A) = c\lambda^*(A)$ for each subset A of $\mathbf{R}^n$.

Problem 18.12. *Show that an arbitrary collection of pairwise disjoint measurable subsets of $\mathbf{R}$, each of which has positive measure, is at-most countable.*

Solution. Let $\mathcal{C}$ be a collection of pairwise disjoint measurable subsets of $\mathbf{R}$ such that $\lambda(C) > 0$ holds for each $C \in \mathcal{C}$. For each n let

$$\mathcal{C}_n = \{C \in \mathcal{C}\colon\ \lambda(C \cap [-n, n]) \ge \tfrac{1}{n}\},$$

and note that $C = \bigcup_{n=1}^{\infty} C_n$. Now if $C_1, \ldots, C_k \in C_n$, then we have

$$\tfrac{k}{n} \le \sum_{i=1}^{k} \lambda(C_i \cap [-n, n]) = \lambda\Big(\Big(\bigcup_{i=1}^{k} C_i\Big) \cap [-n, n]\Big) \le \lambda([-n, n]) = 2n,$$

and so $k \le 2n^2$ holds. This shows that each C_n is a finite set, and consequently C is at-most countable.

Problem 18.13. *Let G be a proper additive subgroup of $\mathbf{R}^n$. If G is a measurable set, then show that $\lambda(G) = 0$.*

Solution. If $\lambda(G) > 0$, then, by Theorem 18.13, the element zero is an interior point of $G - G$. Since G is an additive group, $G - G = G$ holds, and from this it follows that $G = \mathbf{R}^n$, which is a contradiction.

Problem 18.14. *Let $f: \mathbf{R} \to \mathbf{R}$ be additive (i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$) and Lebesgue measurable. Show that f is continuous—and hence, of the form $f(x) = cx$.*

Solution. Assume $f \ne 0$ and let $\varepsilon > 0$. Since f is an additive function, $nf^{-1}([0, \varepsilon]) = f^{-1}([0, n\varepsilon])$ holds (why?). Thus, if $\lambda(f^{-1}([0, \varepsilon])) = 0$, then

$$\lambda(f^{-1}([-n\varepsilon, 0])) = \lambda(f^{-1}([0, n\varepsilon])) = n\lambda(f^{-1}([0, \varepsilon])) = 0$$

holds for each n, and so $\lambda(f^{-1}([-n\varepsilon, n\varepsilon])) = 0$ for all n. From

$$f^{-1}([-n\varepsilon, n\varepsilon]) \uparrow \mathbf{R},$$

it follows that $\lambda(\mathbf{R}) = 0$, which is impossible. Thus, $\lambda(f^{-1}([0, \varepsilon])) > 0$. Since f is also measurable, there exists (by Theorem 18.13) some $\delta > 0$ with

$$(-\delta, \delta) \subseteq f^{-1}([0, \varepsilon]) - f^{-1}([0, \varepsilon]) = f^{-1}([-\varepsilon, \varepsilon]).$$

That is, $-\delta < x < \delta$ implies $-\varepsilon \le f(x) \le \varepsilon$ so that f is continuous at zero. Now apply Lemma 18.7.

Problem 18.15. *Show that an arbitrary union of proper intervals of $\mathbf{R}$ is a Lebesgue measurable set.*

Solution. Let $\{I_\alpha : \alpha \in A\}$ be a family of "proper" intervals (an interval is proper whenever its endpoints a and b satisfy $a < b$) and let $E = \bigcup_{\alpha \in A} I_\alpha$. Write $E = \bigcup_{x \in E} C_x$, where C_x denotes the component of x in E. Since each x belongs to a proper subinterval of E, we see that each C_x is a proper interval; see part (g) Problem 6.11. Since the distinct components C_x are pairwise disjoint,

we see that there are at-most countably many C_x and so E is the union of at-most countably many intervals. Now, use the fact that each interval is a Lebesgue measurable set to infer that E itself is a Lebesgue measurable set.

Problem 18.16. *Let C be a closed nowhere dense subset of $\mathbf{R}^n$ such that $\lambda(C) > 0$. Show that the characteristic function χ_C cannot be continuous on the complement of any Lebesgue null set of $\mathbf{R}^n$. Also, show that χ_C will be continuous on the complement of a properly chosen open set whose Lebesgue measure can be made arbitrarily small.*

Solution. Let $A \subseteq \mathbf{R}^n$ be a Lebesgue null set. Since $\lambda(C) > 0$ and $\lambda(A) = 0$, it follows that $A^c \cap C \neq \emptyset$. Fix some $a \in A^c \cap C$. We claim that $\chi_C\colon A^c \longrightarrow \mathbf{R}$ is not continuous at $x = a$.

Indeed, if $\chi_C\colon A^c \longrightarrow \mathbf{R}$ is continuous at a, then there exists some open ball $B(a,r)$ with $\chi_C(x) = 1$ for all $x \in B(a,r) \cap A^c$; i.e., $B(a,r) \cap A^c \subseteq C$ holds. Since $\lambda(A) = 0$, it follows that $B(a,r) \cap A^c$ is dense in $B(a,r)$, and therefore, $B(a,r) \subseteq C$ (since $B(a,r) \cap A^c \subseteq C$ and C is closed), contradicting the fact that C is nowhere dense.

Now, let $\varepsilon > 0$. By Theorem 18.2, there exists an open set V with $C \subseteq V$ and $\lambda(V \setminus C) < \varepsilon$. Note that the set $O = V \setminus C = V \cap C^c$ is open, and $\lambda(O) < \varepsilon$. We claim that $\chi_C\colon O^c \longrightarrow \mathbf{R}$ is continuous.

To see this, let $a \notin O = V \cap C^c$. We have two cases.
1) $a \in C$. Since $C \subseteq V$, there exists some open ball $B(a,r)$ with $B(a,r) \subseteq V$. Now, note that

$$B(a,r) \cap O^c = B(a,r) \cap [V^c \cup C] = B(a,r) \cap C \subseteq C.$$

Thus, if $x \in B(a,r) \cap O^c$, then $\chi_C(x) = 1$. This shows that the function $\chi_C\colon O^c \longrightarrow \mathbf{R}$ is continuous at $x = a$.
2) $a \in C^c$. Choose an open ball $B(a,r)$ such that $B(a,r) \subseteq C^c$. Then,

$$B(a,r) \cap O^c = B(a,r) \cap [V^c \cup C] = B(a,r) \cap V^c \subseteq V^c \subseteq C^c.$$

Thus, $x \in B(a,r) \cap O^c$ implies $\chi_C(x) = 0$, which shows that in this case $\chi_C\colon O^c \longrightarrow \mathbf{R}$ is continuous at $x = a$.

Problem 18.17. *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a continuous function. Show that the graph*

$$G = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n))\colon (x_1, \ldots, x_n) \in \mathbf{R}^n\}$$

of f has $(n+1)$-dimensional Lebesgue measure zero.

Solution. Denote by λ_{n+1} and λ_n the $(n+1)$-dimensional and n-dimensional Lebesgue measures, respectively. Fix some k and let $A = [-k,k]\times\cdots\times[-k,k]$. Now, let $\varepsilon > 0$. By the uniform continuity of f on A, there exists some $\delta > 0$ such that $x, y \in A$ and $|x_i - y_i| < \delta$ for $1 \le i \le n$ imply $|f(x) - f(y)| < \varepsilon$. Fix a partition P of $[-k,k]$ with mesh $|P| < \delta$, and let $Q = P \times \cdots \times P$. Then, Q subdivides A into a finite number of distinct closed cells, say $A_1, \ldots, A_p$. (Note that the open cells corresponding to $A_1, \ldots, A_p$ are pairwise disjoint). For each $1 \le i \le p$ fix some $a_i \in A_i$, and let $I_i = [f(a_i) - \varepsilon, f(a_i) + \varepsilon]$. Then, $G_k \subseteq \bigcup_{i=1}^{p}(A_i \times I_i)$ holds, and so

$$\lambda_{n+1}(G_k) \le \sum_{i=1}^{p} \lambda_{n+1}(A_i \times I_i) = \sum_{i=1}^{p} \lambda_n(A_i)\cdot 2\varepsilon = (2k)^n \cdot 2\varepsilon$$

holds for all $\varepsilon > 0$. This shows that $\lambda_{n+1}(G_k) = 0$ for each k. To complete the proof, now apply Theorem 15.4 to $G_k \uparrow G$.

Problem 18.18. *Let X be a Hausdorff topological space, and let μ be a regular Borel measure on X. Show the following:*

a. *If A is an arbitrary subset of X, then*

$$\mu^*(A) = \inf\{\mu(\mathcal{O})\colon\ \mathcal{O} \text{ open and } A \subseteq \mathcal{O}\}.$$

b. *If A is a measurable subset of X with $\mu^*(A) < \infty$, then*

$$\mu^*(A) = \sup\{\mu(K)\colon\ K \text{ compact and } K \subseteq A\}.$$

c. *If μ is σ-finite and A is a measurable subset of X, then*

$$\mu^*(A) = \sup\{\mu(K)\colon\ K \text{ compact and } K \subseteq A\}.$$

Solution. (a) Since every σ-set is a Borel set, Problem 15.2 shows that

$$\mu^*(A) = \inf\big\{\mu(B)\colon\ B \text{ is a Borel set satisfying } A \subseteq B\big\}.$$

Now, use property (2) of Definition 18.4.
(b) Let A be a measurable set with $\mu^*(A) < \infty$ and let $\varepsilon > 0$. Pick an open set V with $A \subseteq V$ and $\mu^*(V) < \mu^*(A) + \varepsilon$. Similarly, choose an open set W such that $V \setminus A \subseteq W \subseteq V$ and

$$\mu^*(W) < \mu^*(V \setminus A) + \varepsilon = \mu^*(V) - \mu^*(A) + \varepsilon < 2\varepsilon.$$

Next, pick a compact set C such that $C \subseteq V$ and $\mu^*(V) < \mu^*(C) + \varepsilon$. Set $K = C \cap W^c$, and note that K is a compact subset of A. Moreover,

$$\begin{aligned} 0 \le \mu^*(A) - \mu^*(K) &= \mu^*(A \setminus K) \le \mu^*(V \setminus K) \\ &= \mu^*\big((V \setminus C) \cup W\big) \le \big[\mu^*(V) - \mu^*(C)\big] + \mu^*(W) < 3\varepsilon \end{aligned}$$

holds, and the desired conclusion follows.
(c) Straightforward using (b).

Problem 18.19. *If A is a (Lebesgue) measurable subset of $\mathbb{R}$ of positive measure and $0 < \delta < \lambda(A)$, then show that there exists a measurable subset B of A satisfying $\lambda(B) = \delta$.*

Solution. We shall present two solutions. The first one will employ the Axiom of Choice (via Zorn's Lemma); the second one will establish the validity of the conclusion without using the Axiom of Choice and without assuming that A is a measurable set.

(a) Consider a measurable subset A of $\mathbb{R}$ and some $\delta > 0$ satisfying $0 < \delta < \lambda(A)$. Since $\lambda(A \cap [-n, n]) \uparrow \lambda(A)$ holds, replacing A by some $A \cap [-n, n]$, we can assume that $\lambda(A) < \infty$ also holds.

Next, we shall denote by $\mathcal{A}$ the set of all collections $\mathcal{C}$ of pairwise disjoint measurable subsets of A such that:
a) $\lambda(C) > 0$ holds for each $C \in \mathcal{C}$ (and so $\mathcal{C}$ is at most countable); and
b) The Lebesgue measurable set $\bigcup_{C \in \mathcal{C}} C$ satisfies $\lambda\big(\bigcup_{C \in \mathcal{C}} C\big) \le \delta$.

From Problem 15.18, it is easy to see that $\mathcal{A} \neq \emptyset$. Under the inclusion relation $\subseteq$ the set $\mathcal{A}$ is a partially ordered set. We claim that the partially ordered set $(\mathcal{A}, \subseteq)$ satisfies the hypothesis of Zorn's Lemma. To see this, let $\{\mathcal{C}_i\colon i \in I\}$ be a chain of $\mathcal{A}$ (i.e., for each pair $i, j \in I$ either $\mathcal{C}_i \subseteq \mathcal{C}_j$ or $\mathcal{C}_j \subseteq \mathcal{C}_i$ holds true). Our claim, will be established, if we can show that $\mathcal{C} = \bigcup_{i \in I} \mathcal{C}_i \in \mathcal{A}$. Note first that if $B, C \in \mathcal{C}$, then $B, C \in \mathcal{C}_i$ must hold for at least one $i \in I$, and so $B \cap C = \emptyset$. In particular, it follows that $\mathcal{C}$ is at most countable. Now, if $B_1, \ldots, B_k \in \mathcal{C}$, then $B_1, \ldots, B_k \in \mathcal{C}_i$ also must hold for some i (why?), and so $\lambda\big(\bigcup_{r=1}^k B_r\big) \le \lambda\big(\bigcup_{B \in \mathcal{C}_i} B\big) \le \delta$. Since $\mathcal{C}$ is at most countable, it follows that $\lambda\big(\bigcup_{B \in \mathcal{C}} B\big) \le \delta$. That is, $\mathcal{C} \in \mathcal{A}$.

Now, by Zorn's Lemma, the collection $\mathcal{A}$ has a maximal element, say $\mathcal{C}$. If $B = \bigcup_{C \in \mathcal{C}} C$, then we claim that the measurable set B satisfies $\lambda(B) = \delta$ (and this will complete the proof). To see the latter, assume by way of contradiction that $\lambda(B) < \delta$. Then, we have $0 < \eta = \delta - \lambda(B) \le \lambda(A) - \lambda(B) = \lambda(A \setminus B)$ holds, and so by Problem 15.18 there exists a measurable subset D of $A \setminus B$

satisfying $0 < \lambda(D) < \eta$ (clearly, $D \notin C$). In view of $B \cap D = \emptyset$ and

$$\lambda(B \cup D) = \lambda(B) + \lambda(D) < \lambda(B) + \delta - \lambda(B) = \delta,$$

we see that $C_1 = C \cup \{D\} \in \mathcal{A}$. However, this contradicts the maximality property of C, and so $\lambda(B) = \delta$ must hold, as desired.

(b) For this solution the set A is an arbitrary subset of $\mathbf{R}$ satisfying $\lambda(A) > 0$. As in the preceding, we can assume that $A \subseteq [-k, k]$ holds for some k. Now, consider the function $f: [-k, k] \longrightarrow \mathbf{R}$ defined by

$$f(t) = \lambda(A \cap [-k, t]), \quad t \in [-k, k].$$

Clearly, $f(-k) = 0$ and $f(k) = \lambda(A)$. We claim that f is a continuous function. Indeed, if $-k \leq s < t \leq k$, then

$$f(t) = \lambda(A \cap [-k, t]) \leq \lambda(A \cap [-k, s]) + \lambda(A \cap (s, t]) \leq f(s) + t - s.$$

Therefore, $|f(s) - f(t)| \leq |t - s|$ holds for all $s, t \in [-k, k]$ and so f is a continuous function.

Finally, by the Intermediate Value Theorem, there exists some $-k \leq x \leq k$ such that the subset $B = A \cap [-k, x]$ of A (which is measurable if A is measurable) satisfies $f(x) = \lambda(B) = \delta$.

Problem 18.20. *Let E be a Lebesgue measurable subset of $\mathbf{R}$ of finite Lebesgue measure. Show that the function $f_E: \mathbf{R} \to \mathbf{R}$, defined by*

$$f_E(x) = \lambda(E \Delta (x + E)),$$

is uniformly continuous.

Solution. The solution goes by steps.

(1) Assume first that $E = (a, b)$ is a bounded open subinterval of $\mathbf{R}$. In this case, an easy calculation shows that

$$f_E(x) = \begin{cases} 2|x|, & \text{if } |x| < b - a \\ 2(b - a), & \text{if } |x| \geq b - a. \end{cases}$$

This guarantees that f_E is uniformly continuous in this case.

(2) Assume that E and F are two Lebesgue measurable subsets of $\mathbf{R}$ of finite measure such that f_E and f_F are both uniformly continuous. Put $G = E \cup F$. We shall show that f_G is also uniformly continuous.

To see this, notice first that

$$\left|\chi_G(y) - \chi_{x+G}(y)\right| \le \left|\chi_E(y) - \chi_{x+E}(y)\right| + \left|\chi_F(y) - \chi_{x+F}(y)\right|$$

implies

$$\lambda\big(G\Delta(x+G)\big) \le \lambda\big(E\Delta(x+E)\big) + \lambda\big(F\Delta(x+F)\big).$$

Hence,

$$\begin{aligned}
\left|f_G(x) - f_G(y)\right| &= \left|\lambda(G\Delta(x+G)) - \lambda(G\Delta(y+G))\right| \\
&\le \lambda\big([G\Delta(x+G)]\Delta[G\Delta(y+G)]\big) \\
&= \lambda\big((x+G)\Delta(y+G)\big) = \lambda\big(G\Delta(y-x+G)\big) \\
&\le \lambda\big(E\Delta(y-x+E)\big) + \lambda\big(F\Delta(y-x+F)\big) \\
&= f_E(y-x) + f_F(y-x).
\end{aligned}$$

Since f_E and f_F are uniformly continuous, it follows that f_G is likewise uniformly continuous. (Actually, the continuity of f_E and f_F at zero is what is needed here.)

(3) By induction, we can show that if $E = \bigcup_{i=1}^{n} E_i$ with each E_i Lebesgue measurable having finite measure and $E_i \cap E_j = \emptyset$ if $i \neq j$, then f_E is uniformly continuous.

(4) Now, let $\epsilon > 0$. Pick a finite collection of pairwise disjoint bounded open intervals $I_1, \dots, I_n$ such that the set $G = \bigcup_{i=1}^{n} I_i$ satisfies $\lambda(E\Delta G) < \epsilon$. Then, as previously, we have

$$\begin{aligned}
\left|f_E(x) - f_E(y)\right| &= \left|\lambda(E\Delta(x+E)) - \lambda(E\Delta(y+E))\right| \\
&\le \lambda\big(E\Delta(y-x+E)\big) \\
&\le \lambda(E\Delta G) + \lambda(G\Delta(y-x+G)) + \lambda((y-x+G)\Delta(y-x+E)) \\
&< 2\epsilon + \lambda(G\Delta(y-x+G)) \\
&= 2\epsilon + f_G(y-x).
\end{aligned}$$

This easily implies that f_E must be a uniformly continuous function.

19. CONVERGENCE IN MEASURE

Problem 19.1. *Let $\{f_n\}$ be a sequence of measurable functions and let $f: X \to \mathbb{R}$. Assume that* $\lim \mu^*(\{x \in X: |f_n(x) - f(x)| \ge \epsilon\}) = 0$ *holds for every $\epsilon > 0$. Show that f is a measurable function.*

Solution. Pick a sequence $\{k_n\}$ of strictly increasing positive integers such that $\mu^*(\{x \in X: |f_k(x) - f(x)| \geq \frac{1}{n}\}) < 2^{-n}$ for all $k > k_n$. Set

$$E_n = \{x \in X: |f_{k_n}(x) - f(x)| \geq \tfrac{1}{n}\}$$

for each n and let $E = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n$. Then,

$$\mu^*(E) \leq \mu^*\Big(\bigcup_{n=m}^{\infty} E_n\Big) \leq \sum_{n=m}^{\infty} \mu^*(E_n) \leq 2^{1-m}$$

holds for all m, so that $\mu^*(E) = 0$. Also, if $x \notin E$, then there exists some m such that $x \notin \bigcup_{n=m}^{\infty} E_n$, and so $|f_{k_n}(x) - f(x)| < \frac{1}{n}$ holds for each $n \geq m$. Therefore, $\lim f_{k_n}(x) = f(x)$ for each $x \notin E$, and so $f_{k_n} \longrightarrow f$ a.e. holds. The latter (by Theorem 16.6) easily implies that f is a measurable function.

Problem 19.2. *Assume that $\{f_n\} \subseteq \mathcal{M}$ satisfies $f_n \uparrow$ and $f_n \xrightarrow{\mu} f$. Show that $f_n \uparrow f$ a.e. holds.*

Solution. By Theorem 19.4, there exists a subsequence $\{f_{k_n}\}$ of the sequence $\{f_n\}$ with $f_{k_n} \longrightarrow f$ a.e. Since $f_n \uparrow$, it easily follows that $f_n \uparrow f$ a.e. holds.

Problem 19.3. *If $\{f_n\} \subseteq \mathcal{M}$ satisfies $f_n \xrightarrow{\mu} f$ and $f_n \geq 0$ a.e. for each n, then show that $f \geq 0$ a.e. holds.*

Solution. Since, by Theorem 19.4, some subsequence of $\{f_n\}$ converges almost everywhere to f, we must have $f \geq 0$ a.e.

Problem 19.4. *Let $\{f_n\} \subseteq \mathcal{M}$ and $\{g_n\} \subseteq \mathcal{M}$ satisfy $f_n \xrightarrow{\mu} f$, $g_n \xrightarrow{\mu} g$, and $f_n = g_n$ a.e. for each n. Show that $f = g$ a.e. holds.*

Solution. Since $f_n \xrightarrow{\mu} f$ implies $f_{k_n} \xrightarrow{\mu} f$ for each subsequence $\{f_{k_n}\}$ of $\{f_n\}$, by passing to two subsequences (if necessary), we can choose a strictly increasing sequence $\{k_n\}$ of positive integers such that $f_{k_n} \longrightarrow f$ a.e. and $g_{k_n} \longrightarrow g$ a.e. This easily implies $f = g$ a.e.

Problem 19.5. *Let (X, S, μ) be a finite measure space. Assume that two sequences $\{f_n\}$ and $\{g_n\}$ of $\mathcal{M}$ satisfy $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$. Show that $f_n g_n \xrightarrow{\mu} fg$. Is this statement true if $\mu^*(X) = \infty$?*

Solution. By Theorem 19.4, the only possible limit of $\{f_n g_n\}$ is fg. Consequently, if $f_n g_n \xrightarrow{\mu} fg$ does not hold, then there exist $\varepsilon > 0$ and $\delta > 0$ and some subsequence of $\{f_n g_n\}$ (which we shall denote by $\{f_n g_n\}$ again) such that

$$\mu^*\big(\{x \in X\colon |f_n(x)g_n(x) - f(x)g(x)| \geq \varepsilon\}\big) \geq \delta \tag{$\star$}$$

holds for all n. In view of $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$, Theorem 19.4 shows that for some subsequence $\{f_{k_n} g_{k_n}\}$ of $\{f_n g_n\}$ we must have $f_{k_n} g_{k_n} \longrightarrow fg$ a.e. Now, note that (by Theorem 19.5) $f_{k_n} g_{k_n} \xrightarrow{\mu} fg$ holds, contrary to $(\star)$. Thus, $f_n g_n \xrightarrow{\mu} fg$ holds.

If $\mu^*(X) = \infty$, then the conclusion is no longer true. An example: Take $X = (0, \infty)$ with the Lebesgue measure. Consider the functions $f_n(x) = \sqrt{x^4 + \frac{x}{n}}$ and $f(x) = x^2$. Then, $f_n \xrightarrow{\lambda} f$, while $f_n^2 \not\xrightarrow{\lambda} f^2$.

Problem 19.6. *Show that a sequence of measurable functions $\{f_n\}$ on a finite measure space converges to f in measure if and only if every subsequence of $\{f_n\}$ has in turn a subsequence which converges to f a.e.*

Solution. The conclusion follows immediately from Theorems 19.4 and 19.5.

Problem 19.7. *Define a sequence $\{f_n\}$ of $\mathcal{M}$ to be μ-***Cauchy** *whenever for each $\epsilon > 0$ and $\delta > 0$ there exists some k (depending on ϵ and δ) such that $\mu^*(\{x \in X\colon |f_n(x) - f_m(x)| \geq \epsilon\}) < \delta$ holds for all $n, m \geq k$.*

Show that a sequence $\{f_n\}$ of $\mathcal{M}$ is a μ-Cauchy sequence if and only if there exists a measurable function f such that $f_n \xrightarrow{\mu} f$.

Solution. If $f_n \xrightarrow{\mu} f$, then the inclusion

$$\{x\colon |f_n(x) - f_m(x)| \geq 2\varepsilon\} \subseteq \{x\colon |f_n(x) - f(x)| \geq \varepsilon\} \cup \{x\colon |f_m(x) - f(x)| \geq \varepsilon\}$$

easily implies that $\{f_n\}$ is a μ-Cauchy sequence.

For the converse, assume that $\{f_n\}$ is a μ-Cauchy sequence. It suffices to show that $\{f_n\}$ has a subsequence that converges in measure (why?). To this end, start by selecting a subsequence $\{g_n\}$ of $\{f_n\}$ satisfying

$$\mu^*\big(\{x\colon |g_n(x) - g_m(x)| \geq 2^{-n}\}\big) < 2^{-n}$$

for all $m \geq n$. Let $E_n = \{x\colon |g_{n+1}(x) - g_n(x)| \geq 2^{-n}\}$. Also, let

$$F_n = \bigcup_{k=n}^{\infty} E_k = \{x\colon |g_{k+1}(x) - g_k(x)| \geq 2^{-k} \text{ holds for some } k \geq n\}.$$

Clearly, $\mu^*(F_n) \leq \sum_{k=n}^{\infty} \mu^*(E_k) \leq 2^{1-n}$ holds for all n, and hence the measurable set $F = \bigcap_{n=1}^{\infty} F_n$ satisfies $\mu^*(F) = 0$. Now, note for each fixed $x \notin F$ there exists some positive integer k_x such that $x \notin F_n$ holds for all $n \geq k_x$. Thus, for $n \geq k_x$, we have

$$\left|g_{n+p}(x) - g_n(x)\right| \leq \sum_{i=n}^{\infty}\left|g_{i+1}(x) - g_i(x)\right| \leq 2^{1-n}.$$

Therefore, $\{g_n(x)\}$ is a Cauchy sequence of real numbers for each $x \notin F$. Thus, there exists a function $g \in \mathcal{M}$ such that $g_n(x) \to g(x)$ holds for each $x \notin F$.

Now, if $n > k$ and $x \notin F_n$, then

$$\left|g_{n+1}(x) - g_{n+p}(x)\right| \leq \sum_{i=n+1}^{\infty} \left|g_{i+1} - g_i(x)\right| \leq 2^{-n}$$

implies that $|g_{n+1}(x) - g(x)| \leq 2^{-n} < 2^{-k}$ for all $n > k$. Thus,

$$\{x \in X\colon |g_{n+1}(x) - g(x)| \geq 2^{-k}\} \subseteq F_n$$

holds for all $n > k$. Finally, to see that $g_n \xrightarrow{\mu} g$ holds, note that for $n > k$, we have

$$\begin{aligned}
&\{x \in X\colon \left|g_n(x) - g(x)\right| \geq 2^{1-k}\} \\
&\subseteq \{x \in X\colon \left|g_n(x) - g_{n+1}(x)\right| \geq 2^{-k}\} \cup \{x \in X\colon \left|g_{n+1}(x) - g(x)\right| \geq 2^{-k}\} \\
&\subseteq E_n \cup F_n = F_n.
\end{aligned}$$

20. ABSTRACT MEASURABILITY

Problem 20.1. *Let $\mathcal{R}$ be a nonempty collection of subsets of a set X. Show that $\mathcal{R}$ is a ring if and only if $\mathcal{R}$ is closed under symmetric differences and finite intersections.*

Solution. Assume first that the nonempty collection $\mathcal{R}$ is a ring. That is, assume that $A, B \in \mathcal{R}$ imply $A \cup B \in \mathcal{R}$ and $A \setminus B \in \mathcal{R}$. Then the identities

$$A \Delta B = (A \setminus B) \cup (B \setminus A) \quad \text{and} \quad A \cap B = A \setminus (A \setminus B)$$

easily imply that $\mathcal{R}$ is closed under symmetric differences and finite intersections.

For the converse assume that $\mathcal{R}$ is closed under symmetric differences and finite intersections. Then, the identities

$$A \setminus B = A \Delta (A \cap B) \quad \text{and} \quad A \cup B = (A \Delta B) \Delta (A \cap B)$$

guarantee that $\mathcal{R}$ is a ring.

Problem 20.2. *If $\mathcal{R}$ is a ring of subsets of a set X, then show that the collection*

$$\mathcal{A} = \{A \subseteq X\colon \text{ Either } A \text{ or } A^c \text{ belongs to } \mathcal{R}\}$$

is an algebra of sets.

Solution. From the definition of $\mathcal{A}$, it easily follows that if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$, i.e., that $\mathcal{A}$ is closed under complementation.

Now, assume that $A, B \in \mathcal{A}$. If $A, B \in \mathcal{R}$, then since $\mathcal{R}$ (as being a ring) is closed under finite unions, we have $A \cup B \in \mathcal{R}$ and so $A \cup B \in \mathcal{A}$. If $A^c, B^c \in \mathcal{R}$, then $A^c \setminus (A^c \setminus B^c) \in \mathcal{R}$, and so

$$A \cup B = (A^c \cap B^c)^c = \left[A^c \setminus (A^c \setminus B^c)\right]^c \in \mathcal{A}.$$

Now, assume that $A \in \mathcal{R}$ and $B^c \in \mathcal{R}$. Then, $B^c \setminus A = B^c \cap A^c \in \mathcal{R}$, and consequently (from the definition of $\mathcal{A}$), $A \cup B = (A^c \cap B^c)^c \in \mathcal{A}$. The preceding show that $\mathcal{A}$ is an algebra.

Problem 20.3. *In the implication scheme of Figure 3.3 show that no other implication is true by verifying the following regarding an uncountable set X.*

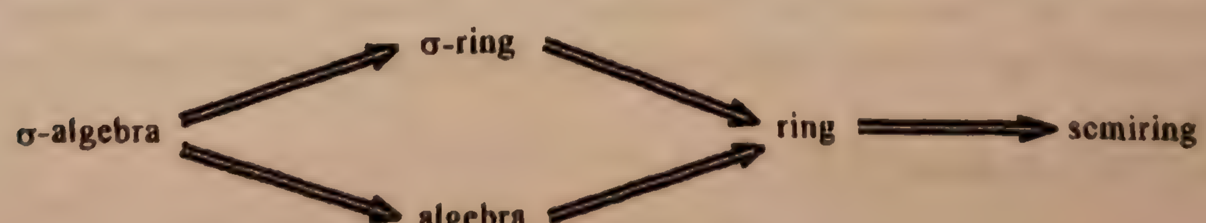

FIGURE 3.3.

a. *The collection of all singleton subsets of X together with the empty set is a semiring but not a ring.*
b. *The collection of all finite subsets of X is a ring but is neither an algebra nor a σ-ring.*
c. *The collection of all subsets of X that are either finite or have finite complement is an algebra but is neither a σ-algebra nor a σ-ring.*
d. *The collection of all at-most countable subsets of X is a σ-ring but not an algebra.*
e. *The collection of all subsets of X that are either at-most countable or have at-most a countable complement is a σ-algebra (which is, in fact, the σ-algebra generated by the singletons).*

Solution. (a) If A and B are singletons, then $A \cap B$ and $A \setminus B$ are either empty or singletons. This shows that the collection of all singletons together with the empty set is a semiring. However, it should be obvious that finite unions of singletons need not be a singleton, and so the collection of all singletons is not an algebra.

(b) Let $\mathcal{R}$ denote the collection of all finite subsets of (the infinite) set X. If $A, B \in \mathcal{R}$, then $A \cup B$ and $A \setminus B$ are finite sets and so $A \cup B$ and $A \setminus B$ belong to $\mathcal{R}$. This shows that $\mathcal{R}$ is a ring. Since the complement of a finite set is infinite, it follows that $\mathcal{R}$ is not closed under complementation, and so $\mathcal{R}$ is not an algebra.

To see that $\mathcal{R}$ is not a σ-ring, let $A = \{a_1, a_2, \ldots\}$ be a countable subset of X, and for each n let $A_n = \{a_n\} \in \mathcal{R}$. Then, $\bigcup_{n=1}^{\infty} A_n = A \notin \mathcal{R}$, and this shows that $\mathcal{R}$ is not a σ-ring.

(c) If $\mathcal{R}$ is the ring of all finite subsets, then by part (b) the collection

$$\mathcal{A} = \{A \subseteq X\colon \text{ Either } A \text{ or } A^c \text{ belongs to } \mathcal{R}\}$$

is an algebra of sets. To see that $\mathcal{A}$ is not a σ-ring (and hence neither a σ-algebra), let $A = \{a_1, a_2, \ldots\}$ be a countable subset of X such that $X \setminus A$ is an infinite set. Clearly, $A \notin \mathcal{A}$. On the other hand, if $A_n = \{a_n\}$, then $A_n \in \mathcal{A}$ and $\bigcup_{n=1}^{\infty} A_n = A \notin \mathcal{A}$. This shows that $\mathcal{A}$ is not a σ-ring.

(d) Let $\mathcal{C}$ denote the collection of all at-most countable subsets of X. Clearly, $A, B \in \mathcal{C}$ imply $A \setminus B \in \mathcal{C}$. Also, $\mathcal{C}$ is closed under countable unions (recall that an at-most countable union of sets each of which is at-most countable is at-most countable; see Theorem 2.6). Therefore, $\mathcal{C}$ is a σ-ring. However, when X is an uncountable set, $\mathcal{C}$ is not closed under complementation, and hence it cannot be an algebra.

(e) This is Problem 12.7.

Problem 20.4. *Show that a Dynkin system is a σ-algebra if and only if it is closed under finite intersections.*

Solution. Let $\mathcal{D}$ be a Dynkin system that is closed under finite intersections. Since $\mathcal{D}$ is also closed under complementation, it is easy to see (by using the identity $A \cup B = (A^c \cap B^c)^c$) that $\mathcal{D}$ is in fact an algebra. So, if $A = \bigcup_{n=1}^{\infty} A_n$ with $\{A_n\} \subseteq \mathcal{D}$, then by letting $B_n = \bigcup_{k=1}^{n} A_k \in \mathcal{D}$, and noting that $B_n \uparrow A$, we see that $A \in \mathcal{D}$. In other words, $\mathcal{D}$ is a σ-algebra.

Problem 20.5. *Give an example of a Dynkin system which is not an algebra.*

Solution. Consider the set $X = \{1, 2, 3, 4\}$, and let

$$\mathcal{D} = \big\{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, X\big\}.$$

Then $\mathcal{D}$ is a Dynkin system (why?), which (since $\{1, 2\} \cup \{1, 3\} = \{1, 2, 3\}$ does not belong to $\mathcal{D}$) fails to be an algebra.

Problem 20.6. *A* **monotone class** *of sets is a family $\mathcal{M}$ of subsets of a set X such that if a sequence $\{A_n\}$ of $\mathcal{M}$ satisfies $A_n \uparrow A$ or $A_n \downarrow A$, then $A \in \mathcal{M}$. Establish the following properties regarding monotone classes:*

a. *We have the following implications:*

$$\sigma\text{-}algebra \Longrightarrow Dynkin\ system \Longrightarrow monotone\ class$$

Give examples to show that no other implication in the preceding scheme is true.

b. *An algebra is a monotone class if and only if it is a σ-algebra.*

c. *The σ-algebra $\sigma(\mathcal{A})$ generated by an algebra $\mathcal{A}$ is the smallest monotone class containing $\mathcal{A}$.*

Solution. (a) The implication scheme follows immediately from the definitions of the three classes of sets involved. An example of a Dynkin system which is not an algebra was exhibited in the preceding problem. Now, if $X = \{1, 2\}$, then the collection $\mathcal{M} = \big\{X, \{1\}\big\}$ is a monotone class but not a Dynkin system.

(b) Let $\mathcal{A}$ be an algebra of sets. If $\mathcal{A}$ is a σ-algebra, then it is clearly a monotone class. For the converse assume that the algebra $\mathcal{A}$ is a monotone class.

Assume $\{A_n\} \subseteq \mathcal{A}$ and put $A = \bigcup_{n=1}^{\infty} A_n$. Let $B_n = \bigcup_{k=1}^{n} A_k \in \mathcal{A}$ and note that $B_n \uparrow A$. Since $\mathcal{A}$ is a monotone class, it follows that $A \in \mathcal{A}$, and so $\mathcal{A}$ is a σ-algebra.

(c) Let $\mathcal{A}$ be an algebra of sets and let $\mathcal{M}$ be the smallest monotone class that contains $\mathcal{A}$, i.e., $\mathcal{M}$ is the intersection of the collection of all monotone classes that include $\mathcal{A}$. Clearly, $\mathcal{A} \subseteq \mathcal{M} \subseteq \sigma(\mathcal{A})$.

Let $\mathcal{C} = \{B \in \mathcal{M}\colon B \setminus A \in \mathcal{M} \text{ for each } A \in \mathcal{A}\}$. An easy verification shows that $\mathcal{C}$ is a monotone class that includes $\mathcal{A}$, and so $\mathcal{M} = \mathcal{C}$. Now, let

$$\mathcal{D} = \{B \in \mathcal{M}\colon M \setminus B \in \mathcal{M} \text{ for each } M \in \mathcal{M}\}.$$

Again, $\mathcal{D}$ is a monotone class which (in view of $\mathcal{M} = \mathcal{C}$) satisfies $\mathcal{A} \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \mathcal{M}$. This shows that $\mathcal{M}$ is, in fact, a Dynkin system. By Dynkin's Lemma 20.8, $\sigma(A) \subseteq \mathcal{M}$, and so $\mathcal{M} = \sigma(A)$.

Problem 20.7. *Show that if X and Y are two separable metric spaces, then $\mathcal{B}_{X\times Y} = \mathcal{B}_X \otimes \mathcal{B}_Y$.*

Solution. Assume that X and Y are two arbitrary topological spaces. For each subset A of X, let

$$\Sigma_A = \{B \subseteq Y\colon A \times B \in \mathcal{B}_{X\times Y}\}.$$

From the identities $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$, we see that if $B, C \in \Sigma_A$, then $B \setminus C \in \Sigma_A$. From $A \times \left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty}(A \times B_n)$, it follows that Σ_A is closed under countable intersections. Observing that $\emptyset \in \Sigma_A$, we infer that Σ_A is a σ-ring. Clearly, Σ_A is a σ-algebra if and only if $Y \in \Sigma_A$.

Next, note that for any open subset $\mathcal{O}$ of X, $V \in \Sigma_{\mathcal{O}}$ for every open subset V of Y. Since Y is itself open, if $\mathcal{O}$ is open, then $\Sigma_{\mathcal{O}}$ is a σ-algebra of subsets of Y that includes the open subsets of Y. Thus, $\mathcal{B}_Y \subseteq \Sigma_{\mathcal{O}}$ for each open subset $\mathcal{O}$ of X. Now, let

$$\mathcal{A} = \{A \subseteq X\colon \mathcal{B}_Y \subseteq \Sigma_A\}.$$

As we have just noticed, $V \in \mathcal{A}$ holds for each open subset V of X. Since (as easily checked) $\Sigma_A = \Sigma_{A^c}$ for each $A \in \mathcal{A}$, we see that $\mathcal{A}$ is closed under complementation. Moreover, if $\{A_n\} \subseteq \mathcal{A}$, then for any Borel subset B of Y, we have $A_n \times B \in \mathcal{B}_{X\times Y}$ for each n. Thus, in view of $\bigcap_{n=1}^{\infty}(A_n \times B) = \left(\bigcap_{n=1}^{\infty} A_n\right) \times B$, we obtain $B \in \Sigma_{\bigcap_{n=1}^{\infty} A_n}$. In other words, $\mathcal{A}$ is closed under countable intersections, and so $\mathcal{A}$ is a σ-algebra including the open subsets of X. This implies $\mathcal{B}_X \subseteq \mathcal{A}$.

We have just established the following: If A is a Borel subset of X and B is a Borel subset of Y, then $A \times B \in \mathcal{B}_{X\times Y}$. Therefore,

$$\mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X\times Y}.$$

For the reverse inclusion, assume that X and Y are two separable metric spaces. Then every open subset of $X \times Y$ is an at-most countable union of sets of the form

$V \times U$, where V is an open subset of X and U an open subset of Y. Consequently, $\mathcal{B}_{X\times Y} \subseteq \mathcal{B}_X \otimes \mathcal{B}_Y$, from which it follows that $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X\times Y}$.

Problem 20.8. *Show that the composition function of two measurable functions is measurable.*

Solution. Assume that $(X, \Sigma_1) \xrightarrow{f} (Y, \Sigma_2) \xrightarrow{g} (Z, \Sigma_3)$ are measurable functions. If $A \in \Sigma_3$, then $g^{-1}(A) \in \Sigma_2$, and so $(g \circ f)^{-1}(A) = f^{-1}\big(g^{-1}(A)\big) \in \Sigma_1$. This shows that $g \circ f$ is measurable.

Problem 20.9. *If (X, Σ) is a measurable space, then show that*

a. *the collection of all real-valued measurable functions defined on X is a function space and an algebra of functions, and*
b. *any real-valued function on X which is the pointwise limit of a sequence of $(\Sigma, \mathcal{B})$-measurable real-valued functions is itself $(\Sigma, \mathcal{B})$-measurable.*

Solution. Repeat the solution of Problem 17.17.

Problem 20.10. *Let (X, Σ) be a measurable space. A Σ-***simple function** *is any measurable function $\phi: X \to \mathbb{R}$ having a finite range, i.e, if ϕ has finite range and its standard representation $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ satisfies $A_i \in \Sigma$ for each i.*

Show that a function $f: X \to [0, \infty)$ is measurable if and only if there exists a sequence $\{\phi_n\}$ of Σ-simple functions such that $\phi_n(x) \uparrow f(x)$ holds for each $x \in X$.

Solution. The proof is identical to the proof of Theorem 17.7. Here it is.

For each n let $A_n^i = \{x \in X\colon (i-1)2^{-n} \le f(x) < i2^{-n}\}$ for $i = 1, 2, \dots, n2^n$, and note that $A_n^i \cap A_n^j = \emptyset$ if $i \neq j$. Since f is measurable, all the A_n^i belong to Σ.

Now, for each n define $\phi_n = \sum_{i=1}^{n2^n} 2^{-n}(i-1)\chi_{A_n^i}$, and note that $\{\phi_n\}$ is a sequence of Σ-simple functions. Also, an easy verification shows that $0 \le \phi_n(x) \le \phi_{n+1}(x) \le f(x)$ holds for all x and all n. Moreover, if x is fixed, then $0 \le f(x) - \phi_n(x) \le 2^{-n}$ holds for all sufficiently large n. This implies $\phi_n(x) \uparrow f(x)$ for all $x \in X$.

Problem 20.11. *Use Corollary 20.10 to show that if a measure μ is σ-finite, then μ^* is the one and only extension of μ to a measure on $\sigma(\mathcal{S})$.*

Solution. Let $\nu\colon \sigma(\mathcal{S}) \to [0, \infty]$ be a measure satisfying $\nu(A) = \mu(A)$ for each $A \in \mathcal{S}$. We shall establish that $\nu(A) = \mu^*(A)$ for each $A \in \sigma(\mathcal{S})$.

Fix $E \in \mathcal{S}$ with $\mu(E) < \infty$ and let

$$\mathcal{S}_E = \{E \cap A\colon\ A \in \mathcal{S}\} = \{B \in \mathcal{S}\colon\ B \subseteq E\}.$$

Clearly, $\mathcal{S}_E$ is a semiring of subsets of E and μ restricted to E is a measure. Moreover, we know (see Problem 15.7) that the outer measure generated by the measure space $(E, \mathcal{S}_E, \mu)$ is simply the restriction of μ^* to $\mathcal{P}(E)$. In addition, we claim that if $\sigma(\mathcal{S}_E)$ denotes the σ-algebra generated by $\mathcal{S}_E$ in $\mathcal{P}(E)$, then

$$\sigma(\mathcal{S}_E) = \{A \cap E\colon\ A \in \sigma(\mathcal{S})\} = \{B \in \sigma(\mathcal{S})\colon\ B \subseteq E\}. \tag{$\star$}$$

To see this, note first that since $\{B \in \sigma(\mathcal{S})\colon\ B \subseteq E\}$ is a σ-algebra containing $\mathcal{S}_E$, we have

$$\sigma(\mathcal{S}_E) \subseteq \{B \in \sigma(\mathcal{S})\colon\ B \subseteq E\}.$$

On the other hand, the collection

$$\mathcal{A} = \{A \subseteq X\colon\ A \cap E \in \sigma(\mathcal{S}_E)\}$$

is a σ-algebra of subsets of X satisfying $\mathcal{S} \subseteq \mathcal{A}$. Hence, $\sigma(\mathcal{S}) \subseteq \mathcal{A}$. In particular, if $B \subseteq E$ satisfies $B \in \sigma(\mathcal{S})$, then $B \in \mathcal{A}$ and so $B = B \cap E \in \sigma(\mathcal{S}_E)$. Consequently, $\{B \in \sigma(\mathcal{S})\colon\ B \subseteq E\} \subseteq \sigma(\mathcal{S}_E)$, and the validity of ($\star$) follows.

Next, note that since $\mathcal{S}_E$ is closed under finite intersections, $\mu^*(E) = \mu(E) = \nu(E) < \infty$, and $\nu(F) = \mu(F) = \mu^*(F)$ for all $F \in \mathcal{S}_E$, it follows from Corollary 20.10 that $\nu(F) = \mu^*(F)$ for all $F \in \sigma(\mathcal{S})$ with $F \subseteq E$.

Now, let $\{E_n\}$ be a pairwise disjoint sequence of $\mathcal{S}$ satisfying $X = \bigcup_{n=1}^\infty E_n$ and $\mu(E_n) < \infty$ for each n. If $A \in \sigma(\mathcal{S})$, then by the preceding discussion we have $\nu(A \cap X_n) = \mu^*(A \cap X_n)$ for each n, and so

$$\begin{aligned}
\nu(A) = \nu(A \cap X) &= \nu\Big(\bigcup_{n=1}^\infty A \cap X_n\Big) = \sum_{n=1}^\infty \nu(A \cap X_n) \\
&= \sum_{n=1}^\infty \mu^*(A \cap X_n) = \mu^*\Big(\bigcup_{n=1}^\infty A \cap X_n\Big) \\
&= \mu^*(A \cap X) = \mu^*(A),
\end{aligned}$$

and we are finished. (For more about the extension of μ, see Problem 15.19.)

Problem 20.12. *Show that the uniform limit of a sequence of measurable functions from a measurable space into a metric space is measurable.*

Solution. Let $\{f_n\}$ be a sequence of measurable functions from a measurable space (X, Σ) into a metric space (Y, d). Suppose f is the uniform limit of $\{f_n\}$. That is, assume that for each $\epsilon > 0$ there exists some n_0 such that $d\big(f_n(x), f(x)\big) < \epsilon$ for all $x \in X$ and all $n \geq n_0$. By passing to a subsequence, we can assume that $d\big(f_n(x), f(x)\big) < \frac{1}{n}$ holds for each n and all $x \in X$.

Since the family of closed sets generates the Borel sets of Y, in order to establish the measurability of f, it suffices to prove that $f^{-1}(C) \in \Sigma$ for each closed set C. To this end, let C be a closed subset of Y.

Let $V_n = \big\{y \in Y\colon d(y, C) < \frac{1}{n}\big\}$. We claim that

$$f^{-1}(C) = \bigcap_{n=1}^{\infty} f^{-1}(V_n). \tag{$\star\star$}$$

To see this, assume $x \in f^{-1}(C)$, i.e., let $f(x) \in C$. From $d\big(f_n(x), C\big) \leq d\big(f_n(x), f(x)\big) < \frac{1}{n}$, we get $f_n(x) \in V_n$ or $x \in f_n^{-1}(V_n)$ for each n. Conversely, if $f_n(x) \in V_n$ for each n, then $d\big(f_n(x), C\big) < \frac{1}{n}$ for each n, and so if we pick some $c_n \in C$ with $d\big(f_n(x), c_n\big) < \frac{1}{n}$, then we have

$$d\big(f(x), C\big) \leq d\big(f(x), c_n\big) \leq d\big(f(x), f_n(x)\big) + d\big(f_n(x), c_n\big) < \tfrac{1}{n} + \tfrac{1}{n} = \tfrac{2}{n}$$

for each n. This implies $d\big(f(x), C\big) = 0$. Since C is a closed set, it follows that $f(x) \in C$, or $x \in f^{-1}(C)$.

Next, use the measurability of each f_n and the fact that each V_n is open to obtain that $f_n^{-1}(V_n) \in \Sigma$ for each n. Now, invoke ($\star\star$) to conclude that $f^{-1}(C) \in \Sigma$.

Problem 20.13. *Let $f, g\colon X \to \mathbf{R}$ be two functions and let $\mathcal{B}$ denote the σ-algebra of all Borel sets of $\mathbf{R}$. Show that there exists a Borel measurable function $h\colon \mathbf{R} \to \mathbf{R}$ satisfying $f = h \circ g$ if and only if $f^{-1}(\mathcal{B}) \subseteq g^{-1}(\mathcal{B})$ holds.*

Solution. Assume $f = h \circ g$ holds for some Borel measurable function $h\colon \mathbf{R} \to \mathbf{R}$. Fix $B \in \mathcal{B}$ and note that $h^{-1}(B) \in \mathcal{B}$. Therefore, $f^{-1}(B) = g^{-1}\big(h^{-1}(B)\big) \in g^{-1}(\mathcal{B})$, and so $f^{-1}(\mathcal{B}) \subseteq g^{-1}(\mathcal{B})$ holds.

For the converse, assume $f^{-1}(\mathcal{B}) \subseteq g^{-1}(\mathcal{B})$. The existence of the Borel measurable function h will be established by steps.

Step I: Assume $f = \chi_A$ for some $A \in f^{-1}(\mathcal{B})$.

Since $f^{-1}(\mathcal{B}) \subseteq g^{-1}(\mathcal{B})$ is true, there exists some $B \in \mathcal{B}$ such that $A = g^{-1}(B)$. Let $h = \chi_B$, and note that $h \circ g = f$.

Step II: Let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$ with the A_i pairwise disjoint and $A_i \in f^{-1}(\mathcal{B})$ for each i. For each i choose some $B_i \in g^{-1}(\mathcal{B})$ such that $A_i = g^{-1}(B_i)$. If we consider the Borel step function $h = \sum_{i=1}^{n} a_i \chi_{B_i}$, then it is easy to see that $h \circ g = f$.

Step III: The general case.

The preceding problem applied with $\Sigma = f^{-1}(\mathcal{B})$ guarantees the existence of a sequence $\{\phi_n\}$ of $f^{-1}(\mathcal{B})$-simple functions satisfying $\phi_n(x) \uparrow f(x)$ for each $x \in X$. Now, by Step II, for each n there exists a Borel measurable function $h_n: \mathbf{R} \to \mathbf{R}$ such that $h_n \circ g = \phi_n$. Next, let

$$B = \{x \in \mathbf{R}:\ \lim_{n\to\infty} h_n(x) = h(x) \text{ exists}\}.$$

It follows (as in Problem 16.7) that $B \in \mathcal{B}$ and $h_n(x)\chi_B(x) \to h(x)$ for each $x \in \mathbf{R}$. If we let $h(x) = 0$ for $x \notin B$, then $h: \mathbf{R} \to \mathbf{R}$ is Borel measurable and satisfies $h \circ g = f$.

Problem 20.14. *Let (X, Σ) be a measurable space, Y, Z_1, and Z_2 be separable metric spaces and Ψ a topological space. Now, assume also that the functions $f_i: X \times Y \to Z_i$, $(i = 1, 2)$, are Carathéodory functions and $g: Z_1 \times Z_2 \to \Psi$ is Borel measurable. Show that the function $h: X \times Y \to \Psi$, defined by*

$$h(x, y) = g\big(f_1(x, y), f_2(x, y)\big),$$

is jointly measurable.

Solution. By Theorem 20.15, each $f_i: X \times Y \to Z_i$ is jointly measurable. This implies that the function $F: X \times Y \to Z_1 \times Z_2$, defined by

$$F(x, y) = \big(f_1(x, y), f_2(x, y)\big)$$

is measurable (why?). Since $g: Z_1 \times Z_2 \to \Psi$ is $(\mathcal{B}_{Z_1 \times Z_2}, \mathcal{B}_\Psi)$-measurable and (by Problem 20.7) $\mathcal{B}_{Z_1} \otimes \mathcal{B}_{Z_2} = \mathcal{B}_{Z_1 \times Z_2}$, it follows that $h = g \circ F$ is likewise measurable.

Problem 20.15. *Let (X, Σ) be a measurable space and (Y, d) a separable metric space. Show that a function $f: X \to Y$ is measurable if and only if for each fixed $y \in Y$ the function $x \mapsto d(y, f(x))$, from X into $\mathbf{R}$, is measurable.*

Solution. Let $f: (X, \Sigma) \to Y$ be a function from a measurable space to a separable metric space. For each $y \in Y$ define the function $g_y: X \to \mathbf{R}$ by $g_y(x) = d\big(y, f(x)\big)$. Note that for each $r > 0$ and each $y \in Y$, we have

$$\begin{aligned} f^{-1}\big(B(y, r)\big) &= \{x \in X:\ f(x) \in B(y, r)\} = \{x \in X:\ d\big(y, f(x)\big) < r\} \\ &= \{x \in X:\ g_y(x) < r\} = g_y^{-1}\big((-\infty, r)\big). \end{aligned}$$

Assume that each g_y is measurable. Then, by the preceding identity, we have $f^{-1}(B(y,r)) = g_y^{-1}((-\infty, r)) \in \Sigma$ for each $y \in Y$ and all $r > 0$. Since Y is a separable metric space, every open set can be written as an at-most countable union of open balls, and so $f^{-1}(\mathcal{O}) \in \Sigma$ holds for each open set $\mathcal{O}$. By Theorem 20.6, f is a measurable function.

For the converse, suppose that f is a measurable function and let $y \in Y$. From $g_y^{-1}((-\infty, r)) = f^{-1}(B(y,r))$ if $r > 0$ and $g_y^{-1}((-\infty, r)) = \emptyset$ if $r \le 0$, we easily infer that g_y is a measurable function for each $y \in Y$.

Problem 20.16. *Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, where $\mathcal{S}$ is a σ-algebra. If $f: X \to \mathbb{R}$ is a Λ_μ-measurable function, then show that there exists a $\mathcal{S}$-measurable function $g: X \to \mathbb{R}$ such that $f = g$ a.e.*

Solution. We can assume $f(x) \ge 0$ for each $x \in X$ (otherwise, we apply the arguments below to f^+ and f^- separately). If $f = \chi_A$ for some $A \in \Lambda_\mu$, then an easy argument (using Theorem 15.11) shows that there exists a null set C such that $B = A \cup C \in \mathcal{S}$. So, if $g = \chi_B$, then g is $\mathcal{S}$-measurable and $f = g$ a.e. holds. It follows that if f is a Λ_μ-simple function, then there exists a $\mathcal{S}$-simple function g such that $f = g$ a.e.

Now, we consider the general case. By Problem 20.10 there exists a sequence $\{\phi_n\}$ of Λ_μ-simple functions satisfying $\phi_n(x) \uparrow f(x)$ for each $x \in X$. For each n fix a $\mathcal{S}$-simple function ψ_n such that $\psi_n = \phi_n$ a.e. By Theorem 15.11, for each n there exists a null set $A_n \in \mathcal{S}$ with $\psi_n(x) = \phi_n(x)$ for all $x \notin A_n$. Put $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$, and note that A is a null set. Moreover, we have $\psi_n \chi_{A^c}(x) \uparrow f\chi_{A^c}(x)$ for each x. If $g = f\chi_{A^c}$, then (by Problem 20.9(b)) g is a $\mathcal{S}$-measurable function satisfying $f = g$ a.e.

CHAPTER 4

THE LEBESGUE INTEGRAL

21. UPPER FUNCTIONS

Problem 21.1. *Let L be the collection of all step functions ϕ such that there exist a finite number of members $A_1, \dots, A_n$ of $\mathcal{S}$ all of finite measure and real numbers $a_1, \dots, a_n$ such that $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$. Show that L is a function space. Is L an algebra of functions?*

Solution. Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\psi = \sum_{j=1}^{m} b_j \chi_{B_j}$, where the A_i and B_j belong to $\mathcal{S}$ and they all have finite measure. By Problem 12.14, there exist pairwise disjoint sets $C_1, \dots, C_k$ of $\mathcal{S}$ such that each A_i and each B_j can be written as a union from the C_i. We can assume that $\bigcup_{r=1}^{k} C_r = \left[\bigcup_{i=1}^{n} A_i\right] \cup \left[\bigcup_{j=1}^{m} B_j\right]$. It is easy to see that ϕ and ψ can be written in the form $\phi = \sum_{r=1}^{k} c_r \chi_{C_r}$ and $\psi = \sum_{r=1}^{k} d_r \chi_{C_r}$. Now, everything follows from the equalities:

1. $\alpha\phi + \beta\psi = \sum_{r=1}^{k} (\alpha c_r + \beta d_r)\chi_{C_r}$;
2. $\phi \vee \psi = \sum_{r=1}^{k} (c_r \vee d_r)\chi_{C_r}$ and $\phi \wedge \psi = \sum_{r=1}^{k} (c_r \wedge d_r)\chi_{C_r}$; and
3. $\phi\psi = \sum_{r=1}^{k} c_r d_r \chi_{C_r}$.

Problem 21.2. *Consider the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 0$ if $x \notin (0, 1]$, and $f(x) = \sqrt{n}$ if $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for some n. Show that f is an upper function and that $-f$ is not an upper function.*

Solution. Put $A_k = \left(\frac{1}{k+1}, \frac{1}{k}\right]$ and note that $\lambda(A_k) = \frac{1}{k(k+1)}$. Thus, if we let

$$\phi_n = \sum_{k=1}^{n} \sqrt{k}\chi_{A_k},$$

then $\{\phi_n\}$ is a sequence of step functions satisfying $\phi_n(x) \uparrow f(x)$ for each x. On

the other hand, the relations

$$\int \phi_n \, d\lambda = \sum_{k=1}^{n} \sqrt{k} \cdot \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{\sqrt{k}(k+1)} \le \sum_{k=1}^{\infty} k^{-\frac{3}{2}} < \infty$$

guarantee that f is an upper function.

Since $-f$ is not bounded from below, there is no step function ϕ satisfying $\phi \le -f$. This implies that $-f$ cannot be an upper function.

Problem 21.3. *Compute $\int f \, d\lambda$ for the upper function f of the preceding exercise.*

Solution. We have $\int f \, d\lambda = \sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n}}$.

Problem 21.4. *Verify that every continuous function $f\colon [a,b] \to \mathbb{R}$ is an upper function—with respect to the Lebesgue measure on $[a,b]$.*

Solution. For each n let $P_n = \{x_0, x_1, \ldots, x_{2^n}\}$ be the partition that divides $[a,b]$ into 2^n subintervals all of the same length $(b-a)2^{-n}$; that is, $x_i = a + i(b-a)2^{-n}$. Let $m_i = \min\{f(x)\colon x \in [x_{i-1}, x_i]\}$, and then define

$$\phi_n = \sum_{i=1}^{2^n} m_i \chi_{[x_{i-1}, x_i)}.$$

Clearly, each ϕ_n is a step function. Using the uniform continuity of f, it is not difficult to see that $\phi_n(x) \uparrow f(x)$ holds for all $x \in [a,b)$. On the other hand, if $f(x) \le M$ holds for each x, then $\int \phi_n \, d\lambda \le M(b-a)$ holds for all n, implying that f is an upper function.

Problem 21.5. *Let A be a measurable set, and let f be an upper function. If $\chi_A \le f$ a.e., then show that $\mu^*(A) < \infty$.*

Solution. Choose a sequence $\{\phi_n\}$ of step functions with $\phi_n \uparrow f$ a.e. Then, $\phi_n \wedge \chi_A \uparrow f \wedge \chi_A = \chi_A$ a.e., and so, by Theorem 17.6,

$$\mu^*(A) = \lim_{n\to\infty} \int \phi_n \wedge \chi_A \, d\mu \le \lim_{n\to\infty} \int \phi_n \, d\mu = \int f \, d\mu < \infty.$$

Problem 21.6. *Let f be an upper function, and let A be a measurable set of finite measure such that $a \le f(x) \le b$ holds for each $x \in A$. Then, show that*

a. *$f\chi_A$ is an upper function, and*
b. *$a\mu^*(A) \le \int f\chi_A \, d\mu \le b\mu^*(A)$.*

Solution. (a) Pick a sequence $\{\phi_n\}$ of step functions with $\phi_n \uparrow f$ a.e. For each n define the step function $\psi_n = (\phi_n \chi_A) \wedge b\chi_A$. Then,

$$\int \psi_n \, d\mu \le \int b\chi_A \, d\mu = b\mu^*(A) < \infty.$$

Since $\psi_n \uparrow f\chi_A$ a.e. holds, $f\chi_A$ is an upper function.
(b) Apply the monotone property of the integral (Theorem 21.5) to the inequality $a\chi_A \le f\chi_A \le b\chi_A$.

Problem 21.7. *Let (X, S, μ) be a finite measure space, and let f be a positive measurable function. Show that f is an upper function if and only if there exists a real number M such that $\int \phi \, d\mu \le M$ holds for every step function ϕ with $\phi \le f$ a.e. Also, show that if this is the case, then*

$$\int f \, d\mu = \sup\left\{ \int \phi \, d\mu \colon \phi \text{ is a step function with } \phi \le f \text{ a.e.} \right\}.$$

Solution. If f is an upper function, then by Theorem 21.5 every step function ϕ with $\phi \le f$ a.e. satisfies $\int \phi \, d\mu \le \int f \, d\mu < \infty$.

Conversely, by Theorem 17.7 there exists a sequence $\{\phi_n\}$ of simple functions with $\phi_n \uparrow f$ a.e. Since (X, S, μ) is a finite measure space, we know that each ϕ_n is a step function. If $\int \phi_n \, d\mu \le M$ holds for all n, then this readily implies that f is an upper function.

The last formula is immediate from Theorem 21.5.

Problem 21.8. *Show that every monotone function $f \colon [a, b] \to \mathbb{R}$ is an upper function—with respect to the Lebesgue measure on $[a, b]$.*

Solution. We assume that $f \colon [a, b] \to \mathbb{R}$ is an increasing function. The "decreasing case" can be proven in a similar fashion and is left for the reader.

By Problem 9.8, we know that the set D of all discontinuities of f is an at-most countable set. In particular, $\lambda(D) = 0$. Now, for each n, let P_n be the partition that subdivides $[a, b]$ into 2^n equal subintervals. That is, let $P_n = \{a_0^n, a_1^n, \dots, a_{2^n}^n\}$,

where $a_i^n = a + \frac{b-a}{2^n} i$ for $i = 0, 1, \ldots, 2^n$. Next, for each $1 \le i \le 2^n$ let

$$m_i^n = \inf\{f(x): x \in [a_{i-1}^n, a_i^n]\},$$

and put $\phi_n = \sum_{i=1}^{2^n} m_i^n \chi_{[a_{i-1}^n, a_i^n)}$. Clearly, each ϕ_n is a step function and, in view of the monotonicity of f, we have

$$\phi_n(x) \le \phi_{n+1}(x) \le f(x)$$

for all $x \in [a, b)$. Put $E = D \cup P_1 \cup P_2 \cup P_3 \cup \cdots$ and note that $\lambda(E) = 0$. We shall establish that $\phi_n(x) \uparrow f(x)$ for each $x \in [a, b] \setminus E$.

To this end, fix some $t \in [a, b] \setminus E$ and let $\epsilon > 0$. Since f is continuous at t, there exists some $\delta > 0$ such that

$$x \in [a, b] \text{ and } t - \delta < x < t + \delta \quad \text{imply} \quad f(t) - \epsilon < f(x) < f(t) + \epsilon. \quad (\star)$$

Next, pick some k such that $\frac{b-a}{2^k} < \delta$ for all $n \ge k$, and then choose the subinterval $[x_{i-1}, x_i]$ of P_k such that $t \in (x_{i-1}, x_i)$. From $(\star)$, it easily follows that

$$\phi_k(t) = \inf\{f(x): x \in [x_{i-1}, x_i]\} \ge f(t) - \epsilon.$$

Therefore, $f(t) - \epsilon \le \phi_k(t) \le \phi_n(t) \le f(t)$ holds for all $n \ge k$, and this shows that $\phi_n(t) \uparrow f(t)$, as claimed.

Finally, note that the monotonicity of f guarantees that $m_i^n \le f(b)$ holds for all $1 \le i \le 2^n$. This implies

$$\int \phi_n \, d\lambda = \sum_{i=1}^{2^n} m_i^n (a_i^n - a_{i-1}^n) \le f(b) \sum_{i=1}^{2^n} (a_i^n - a_{i-1}^n) = f(b)(b-a) < \infty$$

for each n, and this establishes that f is an upper function.

22. INTEGRABLE FUNCTIONS

Problem 22.1. *Show by a counterexample that the integrable functions do not form an algebra.*

Solution. Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$ if $x \notin (0, 1]$, and $f(x) = \sqrt{n}$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$ for some n. From Problem 21.2, we know that f is an integrable function. Now, note that f^2 is not an integrable function.

Problem 22.2. *Let X be a nonempty set, and let δ be the Dirac measure on X with respect to the point a (see Example 13.4). Show that every function $f: X \to \mathbb{R}$ is integrable and that $\int f\, d\delta = f(a)$.*

Solution. Note that $f = f(a)\chi_{\{a\}}$ a.e. holds. Consequently, the function f is integrable and $\int f\, d\delta = f(a)\delta(\{a\}) = f(a)$.

Problem 22.3. *Let μ be the counting measure on $\mathbb{N}$ (see Example 13.3). Show that a function $f: \mathbb{N} \to \mathbb{R}$ is integrable if and only if $\sum_{n=1}^{\infty} |f(n)| < \infty$. Also, show that in this case $\int f\, d\mu = \sum_{n=1}^{\infty} f(n)$.*

Solution. Let $f: \mathbb{N} \longrightarrow \mathbb{R}$. Since every function is measurable, f is integrable if and only if both f^+ and f^- are integrable. So, we can assume that $f(k) \geq 0$ holds for each k.

If $\phi_n = \sum_{k=1}^{n} f(k)\chi_{\{k\}}$, then $\{\phi_n\}$ is a sequence of step functions such that $\phi_n(k) \uparrow_n f(k)$ for each k, and

$$\int \phi_n\, d\mu = \sum_{k=1}^{n} f(k) \uparrow_n \sum_{k=1}^{\infty} f(k).$$

This shows that f is integrable if and only if $\sum_{k=1}^{\infty} f(k) < \infty$, and in this case $\int f\, d\mu = \sum_{k=1}^{\infty} f(k)$ holds.

Problem 22.4. *Show that a measurable function f is integrable if and only if $|f|$ is integrable. Give an example of a nonintegrable function whose absolute value is integrable.*

Solution. Apply Theorems 22.2 and 22.6. For a counterexample: Let E be a non-Lebesgue measurable subset of $[0, 1]$ and consider the function $f = \chi_E - \chi_{[0,1]\setminus E}$.

Problem 22.5. *Let f be an integrable function, and let $\{E_n\}$ be a sequence of disjoint measurable subsets of X. If $E = \bigcup_{n=1}^{\infty} E_n$, then show that*

$$\int_E f\, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f\, d\mu.$$

Solution. Let $F_n = \bigcup_{i=1}^{n} E_i$. Clearly, $|f\chi_{F_n}| \leq |f|$ for each n and $f\chi_{F_n} \longrightarrow f\chi_E$

a.e. Thus, by the Lebesgue Dominated Convergence Theorem, we have

$$\int_E f\,d\mu = \int f\chi_E\,d\mu = \lim_{n\to\infty}\int f\chi_{F_n}\,d\mu$$
$$= \lim_{n\to\infty}\Bigl(\sum_{i=1}^{n}\int f\chi_{E_i}\,d\mu\Bigr) = \sum_{n=1}^{\infty}\int_{E_n} f\,d\mu.$$

Problem 22.6. *Let f be an integrable function. Show that for each $\epsilon > 0$ there exists some $\delta > 0$ (depending on ϵ) such that $|\int_E f\,d\mu| < \epsilon$ holds for all measurable sets with $\mu^*(E) < \delta$.*

Solution. Consider an integrable function f and let $\varepsilon > 0$. From $0 \le |f| \wedge n \uparrow |f|$ and the Lebesgue Dominated Convergence Theorem we get $\int |f| \wedge n\,d\mu \uparrow \int |f|\,d\mu$. So, there exists some n_0 such that $\int(|f| - |f| \wedge n_0)\,d\mu < \frac{\varepsilon}{2}$ for all $n \ge n_0$. Now, put $\delta = \frac{\varepsilon}{2n_0}$ and note that if a measurable set E satisfies $\mu^*(E) < \delta$, then

$$\Bigl|\int_E f\,d\mu\Bigr| = \int_E |f|\,d\mu = \int_E (|f| - |f| \wedge n_0)\,d\mu + \int_E |f| \wedge n_0\,d\mu$$
$$\le \int (|f| - |f| \wedge n_0)\,d\mu + \int_E n_0\,d\mu$$
$$< \tfrac{\varepsilon}{2} + n_0\mu^*(E) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon,$$

as desired.

Problem 22.7. *Show that for every integrable function f the set*

$$\{x \in X : f(x) \ne 0\}$$

*can be written as a countable union of measurable sets of finite measure—referred to as a σ-**finite set**.*

Solution. Each $E_n = \{x \in X : |f(x)| \ge \frac{1}{n}\}$ is a measurable set and, by Theorem 22.5, $\mu^*(E_n) < \infty$ holds. Now, observe that

$$\{x \in X : f(x) \ne 0\} = \bigcup_{n=1}^{\infty} E_n.$$

Problem 22.8. *Let $f:\mathbf{R} \to \mathbf{R}$ be integrable with respect to the Lebesgue measure. Show that the function $g:[0,\infty) \to \mathbf{R}$ defined by*

$$g(t) = \sup\Bigl\{\int |f(x+y) - f(x)|\,d\lambda(x) : |y| \le t\Bigr\}$$

for $t \ge 0$ is continuous at $t = 0$.

Solution. Let $f:\mathbf{R} \longrightarrow \mathbf{R}$ be an upper function and let $\{\phi_n\}$ be a sequence of step functions with $\phi_n \uparrow f$ a.e. Fix some y, and note that $\phi_n(x+y) \uparrow f(x+y)$ holds for almost all x. Since $\chi_A(x+y) = \chi_{A-y}(x)$ and $\lambda(A) = \lambda(A-y)$, it follows that $f(x+y)$ as a function of x is integrable and $\int f(x+y)\,d\lambda(x) = \int f\,d\lambda$. Thus, if f is integrable, then $f(x+y)$ is integrable with respect to x for each fixed y and $\int f(x+y)\,d\lambda(x) = \int f\,d\lambda$ holds. In particular, for each fixed y we have $\int |f(x+y) - f(x)|\,d\lambda(x) \le \int |f(x+y)|\,d\lambda(x) + \int |f(x)|\,d\lambda(x) = 2\int |f|\,d\lambda < \infty$.

Now, for each integrable function $f:\mathbf{R} \longrightarrow \mathbf{R}$ and each $t \ge 0$, define

$$g_f(t) = \sup\Big\{\int \big|f(x+y) - f(x)\big|\,d\lambda(x):\ |y| \le t\Big\} \ge 0.$$

Then, we have

$$g_{f+h}(t) \le g_f(t) + g_h(t) \quad \text{and} \quad g_{\alpha f}(t) = |\alpha| g_f(t).$$

These relations show that the set

$$L = \big\{f:\ f \text{ is integrable and } g_f \text{ is continuous at zero}\big\}$$

is a vector space. Moreover, L has the following approximation property:

- *If f is an integrable function such that for each $\varepsilon > 0$ there exists some $h \in L$ with $\int |f - h|\,d\lambda < \varepsilon$, then $f \in L$.*

Indeed, if f is such a function and $\varepsilon > 0$ is given, then choose $h \in L$ with $\int |f - h|\,d\lambda < \varepsilon$. Pick some $\delta > 0$ with $g_h(t) < \varepsilon$ whenever $0 < t < \delta$, and note that for $|y| \le t$ we have

$$\begin{aligned}
&\int \big|f(x+y) - f(x)\big|\,d\lambda(x)\\
&\le \int \big|f(x+y) - h(x+y)\big|\,d\lambda(x) + \int \big|h(x+y) - h(x)\big|\,d\lambda(x)\\
&\qquad + \int \big|h - f\big|\,d\lambda < 3\varepsilon.
\end{aligned}$$

Thus, $g_f(t) \le 3\varepsilon$ holds for all $0 < t < \delta$ so that $f \in L$.

Now, assume that $f = \chi_{[a,b)}$. If $0 < t < b - a$, then for $|y| \le t$ we have

$$\begin{aligned}
\int \big|f(x+y) - f(x)\big|\,d\lambda(x) &= \int \big|\chi_{[a-y,b-y)}(x) - \chi_{[a,b)}(x)\big|\,d\lambda(x)\\
&= \lambda\big([a-y, b-y)\Delta[a,b)\big) = 2|y| \le 2t,
\end{aligned}$$

and so $g_f(t) \leq 2t$ holds for all $0 < t < b-a$, i.e., $f \in L$. By the approximation property, we have $\chi_A \in L$ for every σ-set A of finite measure, and hence, by the same property, $\chi_A \in L$ for every $A \in \Lambda$ with $\lambda(A) < \infty$ (see Problem 15.2). It follows that L contains the step functions. Since for every integrable function f and each $\varepsilon > 0$ there exists a step function ϕ with $\int |f - \phi|\, d\lambda < \varepsilon$, we infer that L consists of all the integrable functions.

Note. We basically verified here that the collection L satisfies properties (1), (2), and (3) of Theorem 22.12. This guarantees that L coincides with the vector space of all integrable functions.

Problem 22.9. *Let g be an integrable function and let $\{f_n\}$ be a sequence of integrable functions such that $|f_n| \leq g$ a.e. holds for all n. Show that if $f_n \xrightarrow{\mu} f$, then f is an integrable function and $\lim \int |f_n - f|\, d\mu = 0$ holds.*

Solution. By Theorem 19.4, the sequence $\{f_n\}$ has a subsequence that converges to f a.e., and so $|f| \leq g$ a.e. Thus, by Theorem 22.6, the function f is integrable.

Assume that for some $\varepsilon > 0$ there exists a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\int |g_n - f|\, d\mu \geq \varepsilon$. By Theorem 19.4, there exists a subsequence $\{h_n\}$ of $\{g_n\}$ with $h_n \longrightarrow f$ a.e. Now, note that the Lebesgue Dominated Convergence Theorem implies $0 < \varepsilon \leq \int |h_n - f|\, d\mu \longrightarrow 0$, which is impossible. Hence, $\lim \int |f_n - f|\, d\mu = 0$.

Problem 22.10. *Establish the following generalization of Theorem 22.9: If $\{f_n\}$ is a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n|\, d\mu < \infty$, then $\sum_{n=1}^{\infty} f_n$ defines an integrable function and*

$$\int \Big(\sum_{n=1}^{\infty} f_n\Big)\, d\mu = \sum_{n=1}^{\infty} \int f_n\, d\mu.$$

Solution. By Theorem 22.9, the series $g = \sum_{n=1}^{\infty} |f_n|$ defines an integrable function and $\left|\sum_{n=1}^{k} f_n\right| \leq g$ a.e. holds for each k. Since $\sum_{n=1}^{\infty} f_n$ is convergent for almost all points, it follows from the Lebesgue Dominated Convergence Theorem that $\sum_{n=1}^{\infty} f_n$ defines an integrable function and that

$$\int \Big(\sum_{n=1}^{\infty} f_n\Big)\, d\mu = \sum_{n=1}^{\infty} \int f_n\, d\mu.$$

Problem 22.11. *Let f be a positive (a.e.) measurable function, and let*

$$e_i = \mu^*\big(\{x \in X\colon 2^{i-1} < f(x) \leq 2^i\}\big)$$

for each integer i. Show that f is integrable if and only if $\sum_{i=-\infty}^{\infty} 2^i e_i < \infty$.

Solution. Let $E_i = \{x \in X\colon\ 2^{i-1} < f(x) \le 2^i\}$, $i = 0, \pm 1, \pm 2, \ldots$. For each n let $\phi_n = \sum_{i=-n}^{n} 2^i \chi_{E_i}$. Then, there exists some function g with $\phi_n \uparrow g$ a.e. Clearly, g is a measurable function and $0 \le f \le g$ a.e. holds.

Assume that f is integrable. Then, each ϕ_n is a step function, and in view of $\phi_n \le 2f$ (why?), it follows that

$$\sum_{i=-\infty}^{\infty} 2^i e_i = \lim_{n\to\infty} \int \phi_n \, d\mu \le 2\int f \, d\mu < \infty.$$

On the other hand, if $\sum_{i=-\infty}^{\infty} 2^i e_i < \infty$, then each ϕ_n is a step function, and so g is integrable. Since $0 \le f \le g$, Theorem 22.6 shows that f is also integrable.

Problem 22.12. *Let $\{f_n\}$ be a sequence of integrable functions satisfying $0 \le f_{n+1} \le f_n$ a.e. for each n. Then, show that $f_n \downarrow 0$ a.e. holds if and only if $\int f_n \, d\mu \downarrow 0$.*

Solution. Assume $\int f_n d\mu \downarrow 0$. Let $f_n \downarrow f$ a.e.; clearly, $f \ge 0$ a.e. It follows that $\int f \, d\mu = 0$, and thus (by Theorem 22.7) $f = 0$ a.e.

Problem 22.13. *Let f be an integrable function such that $f(x) > 0$ holds for almost all x. If A is a measurable set such that $\int_A f \, d\mu = 0$, then show that $\mu^*(A) = 0$.*

Solution. Let A be a measurable set satisfying $\int_A f \, d\mu = 0$. Next, consider the set $B = \{x \in A\colon\ f(x) \le 0\}$, and note that, by our hypothesis, $\mu^*(B) = 0$. Also, for each n put

$$A_n = \{x \in A\colon\ f(x) \ge \tfrac{1}{n}\}.$$

Then, $A = \left(\bigcup_{n=1}^{\infty} A_n\right) \cup B$, and $f\chi_{A_n} \le f\chi_A$ a.e. holds for each n. Thus,

$$0 \le \tfrac{1}{n}\mu^*(A_n) \le \int_{A_n} f \, d\mu \le \int_A f \, d\mu = 0,$$

which shows that $\mu^*(A_n) = 0$ for each n. This easily implies $\mu^*(A) = 0$.

Problem 22.14. *Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $f\colon X \to \mathbf{R}$ be an integrable function satisfying $f(x) > 0$ for almost all x. If $0 < \varepsilon \le \mu^*(X)$,*

then show that

$$\inf\left\{\int_E f\,d\mu:\ E \in \Lambda_\mu \text{ and } \mu^*(E) \geq \varepsilon\right\} > 0.$$

Solution. We can assume that $f(x) > 0$ holds for each $x \in X$. If for some $0 < \varepsilon \leq \mu^*(X)$ we have

$$\inf\left\{\int_E f\,d\mu:\ E \in \Lambda_\mu \text{ and } \mu^*(E) \geq \varepsilon\right\} = 0,$$

then there exists a sequence $\{E_n\}$ of Λ_μ satisfying $\mu^*(E_n) \geq \epsilon$ and $\int_{E_n} f\,d\mu < \frac{1}{2^n}$ for each n. Put $F_n = \bigcup_{k=n}^{\infty} E_k$ and note that:

- *each F_n is measurable;*
- *$F_{n+1} \subseteq F_n$ holds for each n; and*
- *$\mu^*(F_n) \geq \mu^*(E_n) \geq \varepsilon$ holds for each n.*

If $F = \bigcap_{n=1}^{\infty} F_n$, then F is a measurable set and (by Theorem 15.4(2)) we have

$$\mu^*(F) = \lim_{n\to\infty} \mu^*(F_n) \geq \varepsilon > 0. \qquad (\star)$$

From $f\chi_{F_n} \leq \sum_{k=n}^{\infty} f\chi_{E_k}$, we infer that

$$\int f\chi_{F_n}\,d\mu \leq \sum_{k=n}^{\infty} \int f\chi_{E_k}\,d\mu \leq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}},$$

and so $\int f\chi_{F_n} \downarrow 0$. The latter implies $f\chi_{F_n} \downarrow 0$ a.e. (see Problem 22.12), and since $f\chi_{F_n} \downarrow f\chi_F$, we infer that $f\chi_F = 0$ a.e. In view of $f(x) > 0$ for each x, the latter (in view of Problem 22.13) implies $\mu^*(F) = 0$, contrary to $(\star)$, and the desired conclusion follows.

Problem 22.15. *Let f be a positive integrable function. Define $\nu\colon \Lambda \to [0,\infty)$ by $\nu(A) = \int_A f\,d\mu$ for each $A \in \Lambda$. Show that*

a. *(X, Λ, ν) is a measure space.*
b. *If Λ_ν denotes the σ-algebra of all ν-measurable subsets of X, then show that $\Lambda \subseteq \Lambda_\nu$. Give an example for which $\Lambda \neq \Lambda_\nu$.*
c. *If $\mu^*(\{x \in X:\ f(x) = 0\}) = 0$, then show that $\Lambda = \Lambda_\nu$.*
d. *If g is an integrable function with respect to the measure space (X, Λ, ν),*

then show that fg is integrable with respect to the measure space $(X, \mathcal{S}, \mu)$, and that

$$\int g\,d\nu = \int gf\,d\mu.$$

Solution. (a) This part follows immediately from Problem 22.5.
(b) The measure ν has initial domain Λ. Hence, by Theorem 15.3, $\Lambda \subseteq \Lambda_\nu$ holds.

Consider $\mathbb{R}$ with the Lebesgue measure, and let $f = \chi_{(1,2)}$. Since, in this case, $\nu\big([0,1]\big) = 0$, it follows that every subset of $[0,1]$ is a ν-null set (and hence ν-measurable). On the other hand, not every subset of $[0,1]$ is Lebesgue measurable. Thus, $\Lambda \neq \Lambda_\nu$ holds in this case.
(c) First, observe that ν is a finite measure. Now, let $A \in \Lambda_\nu$ with $\nu^*(A) = 0$. By Theorem 15.11, there exists some $B \in \Lambda$ such that $A \subseteq B$ and $\nu(B) = 0$. The relation $\int_B f\,d\mu = \nu(B) = 0$ combined with Problem 22.13, shows that $\mu^*(A) = 0$. Thus, $A \in \Lambda$. Now, if $V \in \Lambda_\nu$, then pick some $W \in \Lambda$ with $V \subseteq W$ and $\nu(W) = \nu^*(V)$ (Theorem 15.11). Note that $\nu^*(W \setminus V) = 0$, and so, by the preceding discussion, $W \setminus V \in \Lambda$. Finally, $V = W \setminus (W \setminus V) \in \Lambda$ holds, which shows that $\Lambda = \Lambda_\nu$.
(d) We shall assume $g(x) \geq 0$ and $f(x) \geq 0$ for all x. Pick a sequence $\{\phi_n\}$ of ν-step functions such that $0 \leq \phi_n \uparrow g$ ν-a.e. Let

$$G = \{x \in X\colon\ f(x) > 0\}.$$

Clearly, $G \in \Lambda$. Since $G^c = \{x \in X\colon\ f(x) = 0\}$, it follows that $\nu\big(G^c\big) = \int_{G^c} f\,d\mu = 0$. Since f is strictly positive on G, the arguments of part (c) show that whenever $A \subseteq G$, we have:
1) If $A \in \Lambda_\nu$, then $A \in \Lambda$, and
2) By Problem 22.13, $\nu^*(A) = 0$ if and only if $\mu^*(A) = 0$ (and in this case $A \in \Lambda$).

In particular, it follows that $\phi_n f \uparrow fg$ μ-a.e. holds. Now, if $A \in \Lambda_\nu$ satisfies $\nu^*(A) < \infty$, then

$$\int \chi_A\,d\nu = \nu(A \cap G) = \int_{A\cap G} f\,d\mu = \int \chi_A f\,d\mu.$$

This implies that if ϕ is a ν-step function, then ϕf is μ-integrable, and that $\int \phi\,d\nu = \int \phi f\,d\mu$ holds. Now, note that $0 \leq \phi_n f \uparrow fg$ μ-a.e. and $\int \phi_n d\nu = \int \phi_n f\,d\mu$, show that fg is μ-integrable and that $\int g\,d\nu = \int gf\,d\mu$ holds.

Problem 22.16. *Let I be an interval of $\mathbf{R}$, and let $f: I \to \mathbf{R}$ be an integrable function with respect to the Lebesgue measure. For a pair of real numbers a and b with $a \neq 0$, let $J = \{(x-b)/a\colon x \in I\}$. Show that the function $g: J \to \mathbf{R}$ defined by $g(x) = f(ax+b)$ for $x \in J$ is integrable and that $\int_I f\,d\lambda = |a| \int_J g\,d\lambda$ holds.*

Solution. Assume first that $f = \chi_A$ for some measurable set $A \subseteq I$. Clearly, $\frac{1}{a}(A-b) \subseteq J$. Thus, in view of the identity, $\chi_A(ax+b) = \chi_{\frac{1}{a}(A-b)}(x)$, it follows from Problem 15.5 that

$$\int_J g\,d\lambda = \frac{1}{|a|}\lambda(A) = \frac{1}{|a|}\int_I f\,d\lambda.$$

Thus, the formula is true for the characteristic function of a measurable set. It follows that it is also true for step functions.

Now, let f be an upper function. Choose a sequence $\{\phi_n\}$ of step functions with $\phi_n \uparrow f$ a.e. on I. If $\psi_n(x) = \phi_n(ax+b)$ for $x \in J$, then ψ_n is a step function on J and $\psi_n \uparrow g$ a.e. holds on J. (Note that if $B \subseteq I$ satisfies $\lambda(B) = 0$, then by Problem 15.5, we have $\lambda\big(\frac{1}{a}(B-b)\big) = \frac{1}{|a|}\lambda(B) = 0$.) Therefore,

$$|a|\int_J g\,d\lambda = |a| \lim_{n\to\infty}\int_J \psi_n\,d\lambda = \lim_{n\to\infty}\int_I \phi_n\,d\lambda = \int_I f\,d\lambda.$$

Thus, the formula holds true for every integrable function f on I.

Problem 22.17. *Let $(X, \mathcal{S}, \mu)$ be a finite measure space. For every pair of measurable functions f and g let*

$$d(f,g) = \int \frac{|f-g|}{1+|f-g|}\,d\mu.$$

a. *Show that $(\mathcal{M}, d)$ is a metric space.*
b. *Show that a sequence $\{f_n\}$ of measurable functions (i.e., $\{f_n\} \subseteq \mathcal{M}$) satisfies $f_n \xrightarrow{\mu} f$ if and only if $\lim d(f_n, f) = 0$.*
c. *Show that $(\mathcal{M}, d)$ is a complete metric space. That is, show that if a sequence $\{f_n\}$ of measurable functions satisfies $d(f_n, f_m) \to 0$ as n, $m \to \infty$, then there exists a measurable function f such that $\lim d(f_n, f) = 0$.*

Solution. (a) We assume that functions equal μ-a.e. are considered identical. Only the triangle inequality needs verification. To this end, let $f, g, h \in \mathcal{M}$. The

triangle inequality follows immediately from the inequality

$$\frac{|f(x)-g(x)|}{1+|f(x)-g(x)|} \le \frac{|f(x)-h(x)|}{1+|f(x)-h(x)|} + \frac{|h(x)-g(x)|}{1+|h(x)-g(x)|}.$$

For details see the solution of Problem 9.11.
(b) Start by observing that for $x \ge 0$ and $\varepsilon > 0$ we have:

$$x \ge \varepsilon \iff \tfrac{x}{1+x} \ge \tfrac{\varepsilon}{1+\varepsilon}.$$

Now, assume that $\lim d(f_n, f) = 0$ holds. Then, the inequality

$$\begin{aligned}\mu^*\big(\{x \in X\colon |f_n(x)-f(x)| \ge \varepsilon\}\big) &= \mu^*\Big(\Big\{x \in X\colon \tfrac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|} \ge \tfrac{\varepsilon}{1+\varepsilon}\Big\}\Big)\\ &\le \tfrac{1+\varepsilon}{\varepsilon} \cdot d(f_n, f)\end{aligned}$$

easily implies that $f_n \xrightarrow{\mu} f$.

For the converse, assume $f_n \xrightarrow{\mu} f$. If $\lim d(f_n, f) \ne 0$, then there exists some $\varepsilon > 0$ and some subsequence $\{g_n\}$ of $\{f_n\}$ with $d(g_n, f) \ge \varepsilon$ for all n. By passing to a subsequence, we can assume that $g_n \longrightarrow f$ a.e. (Theorem 19.4). In view of $\frac{|g_n - f|}{1+|g_n-f|} \le 1$ and the finiteness of the measure space, the Lebesgue Dominated Convergence Theorem yields $0 < \varepsilon \le \lim d(g_n, f) = 0$, which is absurd. Hence, $\lim d(f_n, f) = 0$ holds.
(c) Assume $d(f_n, f_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$. The inequality

$$\mu^*\big(\{x \in X\colon |f_n(x) - f_m(x)| \ge \varepsilon\}\big) \le \tfrac{1+\varepsilon}{\varepsilon} \cdot d(f_n, f_m)$$

shows that $\{f_n\}$ is a μ-Cauchy sequence. Thus, by Problem 19.7, $f_n \xrightarrow{\mu} f$ holds for some f, and by (b) above, $\lim d(f_n, f) = 0$ also holds.

Conversely, if $\lim d(f_n, f) = 0$ holds, then by part (b) above $f_n \xrightarrow{\mu} f$, which implies that $\{f_n\}$ is a μ-Cauchy sequence.

Problem 22.18. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. For each finite interval I let $f_I = \frac{1}{\lambda(I)} \int_I f\, d\lambda$ and $E_I = \{x \in I\colon f(x) > f_I\}$. Show that*

$$\int_I |f - f_I|\, d\lambda = 2\int_{E_I} (f - f_I)\, d\lambda.$$

Solution. We follow the notation of the problem. Start by observing that

$$\begin{aligned}\int_{E_I}(f-f_I)\,d\lambda+\int_{I\setminus E_I}(f-f_I)\,d\lambda &= \int_I (f-f_I)\,d\lambda\\ &= \int_I f\,d\lambda-\int_I f_I\,d\lambda\\ &= \int_I f\,d\lambda-\int_I f\,d\lambda=0.\end{aligned}$$

Consequently,

$$\int_{E_I}(f-f_I)\,d\lambda=\int_{I\setminus E_I}(f_I-f)\,d\lambda.$$

Now, note that

$$\begin{aligned}\int_I|f-f_I|\,d\lambda &= \int_{E_I}|f-f_I|\,d\lambda+\int_{I\setminus E_I}|f-f_I|\,d\lambda\\ &= \int_{E_I}(f-f_I)\,d\lambda+\int_{I\setminus E_I}(f_I-f)\,d\lambda\\ &= \int_{E_I}(f-f_I)\,d\lambda+\int_{E_I}(f-f_I)\,d\lambda=2\int_{E_I}(f-f_I)\,d\lambda.\end{aligned}$$

Problem 22.19. *Let $f\colon[0,\infty)\to\mathbf{R}$ be a Lebesgue integrable function such that $\int_0^t f(x)\,d\lambda(x)=0$ for each $t\geq 0$. Show that $f(x)=0$ holds for almost all x.*

Solution. Start by observing that

$$\int_{[a,b)} f\,d\lambda=\int_{[0,b)} f\,d\lambda-\int_{[0,a)} f\,d\lambda=0$$

holds for each interval $[a,b)$. By Problem 22.5, we see that $\int_A f\,d\lambda=0$ holds for each σ-set A. From Problem 15.2 (and the Lebesgue Dominated Convergence Theorem), we see that $\int_A f\,d\lambda=0$ holds for each Lebesgue measurable subset A of $\mathbf{R}$.

Now, let $X = \{x \in \mathbb{R}\colon\ f(x) > 0\}$ and $Y = \{x \in X\colon\ f(x) < 0\}$. Clearly, X and Y are both Lebesgue measurable sets, and

$$\int_X f\, d\lambda = \int_Y f\, d\lambda = 0.$$

Now, invoke Problem 22.13 to obtain $\lambda(X) = \lambda(Y) = 0$. Therefore, $f(x) = 0$ holds for almost all x.

Problem 22.20. *Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f, f_1, f_2, \ldots$ be non-negative integrable functions such that $f_n \to f$ a.e. and $\lim \int f_n\, d\mu = \int f\, d\mu$. If E is a measurable set, then show that*

$$\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu.$$

Solution. Assume that the integrable functions $f, f_1, f_2, \ldots$ are non-negative satisfying the hypotheses of the problem and let E be a measurable set. Then the functions $f\chi_E, f_1\chi_E, f_2\chi_E, \ldots$ are non-negative and integrable (because $0 \le f\chi_E \le f$ and $0 \le f_n\chi_E \le f_n$) and $f_n\chi_E \longrightarrow f\chi_E$ holds. Using Fatou's Lemma, we see that

$$\int_E f\, d\mu = \int \liminf f_n\chi_E\, d\mu \le \liminf \int f_n\chi_E\, d\mu = \liminf \int_E f_n\, d\mu. \qquad (\star)$$

Similarly, we have

$$\int_{E^c} f\, d\mu \le \liminf \int_{E^c} f_n\, d\mu. \qquad (\star\star)$$

Therefore,

$$\begin{aligned}
\int f\, d\mu = \int_E f\, d\mu + \int_{E^c} f\, d\mu &\le \liminf \int_E f_n\, d\mu + \liminf \int_{E^c} f_n\, d\mu \\
&\le \liminf\Big(\int_E f_n\, d\mu + \int_{E^c} f_n\, d\mu\Big) \\
&= \liminf \int f_n\, d\mu \\
&= \int f\, d\mu,
\end{aligned}$$

where the second inequality holds by virtue of Problem 4.7(b). It follows that

$$\int_E f\,d\mu + \int_{E^c} f\,d\mu = \liminf \int_E f_n\,d\mu + \liminf \int_{E^c} f_n\,d\mu,$$

and from ($\star$) and ($\star\star$), we see that

$$\liminf \int_E f_n\,d\mu = \int_E f\,d\mu.$$

Now, let $\{g_n\}$ be a subsequence of $\{f_n\}$. Then,

$$g_n \longrightarrow f \text{ a.e.} \quad \text{and} \quad \lim_{n\to\infty} \int g_n\,d\mu = \int f\,d\mu.$$

By the preceding conclusion, we infer that

$$\liminf \int_E g_n\,d\mu = \int_E f\,d\mu,$$

and so there exists a subsequence $\{g_{k_n}\}$ of the sequence $\{g_n\}$ such that $\lim \int_E g_{k_n}\,d\mu = \int_E f\,d\mu$.

Thus, we have demonstrated that every subsequence of the bounded sequence of real numbers $\{\int_E f_n\,d\mu\}$ has a convergent subsequence to $\int_E f\,d\mu$. This means that

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu$$

holds; see Problem 4.2.

Problem 22.21. *If a Lebesgue integrable function* $f\colon [0,1] \to \mathbf{R}$ *satisfies* $\int_0^1 x^{2n} f(x)\,d\lambda(x) = 0$ *for each* $n = 0, 1, 2, \ldots$, *then show that* $f = 0$ *a.e.*

Solution. Let an integrable function $f\colon [0,1] \longrightarrow \mathbf{R}$ satisfy

$$\int_0^1 x^{2n} f(x)\,d\lambda(x) = 0 \quad \text{for } n = 0, 1, 2, \ldots.$$

Since the algebra of functions generated by $\{1, x^2\}$ is uniformly dense in $C[0,1]$ (see Problem 11.5), it follows that $\int_0^1 g(x) f(x)\,d\lambda(x) = 0$ holds for all g in

$C[0, 1]$. Consider the two measurable sets

$$E = \{x \in [0, 1]: \ f(x) > 0\} \quad \text{and} \quad F = \{x \in [0, 1]: \ f(x) < 0\}.$$

We have to show that $\lambda(E) = \lambda(F) = 0$. We shall establish that $\lambda(E) = 0$ holds and leave the identical arguments for F to the reader.

Pick a sequence $\{K_n\}$ of compact sets and a sequence $\{O_n\}$ of open sets of $[0, 1]$ satisfying $K_n \subseteq E \subseteq O_n$ for each n, $K_n \uparrow$, $O_n \downarrow$, and $\lambda(E) = \lim \lambda(K_n) = \lim \lambda(O_n)$. (Here we use the regularity of the Lebesgue measure.) For each n there exists (by Theorem 10.8) a continuous function $g_n\colon [0, 1] \longrightarrow [0, 1]$ satisfying $g_n(x) = 1$ for each $x \in K_n$ and $f(x) = 0$ for each $x \notin O_n$. Clearly, $|g_n f| \le |f|$ and $g_n f \longrightarrow f\chi_E$ a.e. By the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^1 g_n(x) f(x)\, d\lambda(x) = \int_0^1 f(x)\chi_E(x)\, d\lambda(x) = \int_E f\, d\lambda.$$

Taking into account that $\int_0^1 g_n(x) f(x)\, d\lambda(x) = 0$ holds for all n, we infer that $\int_E f\, d\lambda = 0$. Now, invoke Problem 22.13 to infer that $\lambda(E) = 0$, as claimed.

Problem 22.22. *For each n consider the partition*

$$\{0, 2^{-n}, 2\cdot 2^{-n}, 3\cdot 2^{-n}, \ldots, (2^n - 1)\cdot 2^{-n}, 1\}$$

of the interval $[0, 1]$ *and define the function* $r_n\colon [0, 1] \to \mathbb{R}$ *by* $r_n(1) = -1$ *and*

$$r_n(x) = (-1)^{k-1} \ \textit{for} \ (k-1)2^{-n} \le x < k2^{-n} \ (k = 1, 2, \ldots, 2^n).$$

a. *Draw the graphs of* r_1 *and* r_2.
b. *Show that if* $f\colon [0, 1] \to \mathbb{R}$ *is a Lebesgue integrable function, then*

$$\lim_{n\to\infty} \int_0^1 r_n(x) f(x)\, d\lambda(x) = 0.$$

Solution. (a) The graphs of r_1 and r_2 are shown in Figure 4.1.
(b) By Theorem 22.12, it suffices (how?) to establish the claim for the case $f = \chi_{[a,b)}$, where $[a, b)$ is a subinterval of $[0, 1]$. Clearly, $\int_0^1 r_n(x)\chi_{[a,b)}\, d\lambda(x) = \int_a^b r_n(x)\, dx$. Therefore, it suffices to show that $\lim \int_a^b r_n(x)\, dx = 0$ holds for each $0 \le a < b \le 1$.

To this end, fix $0 \le a < b \le 1$ and let $\varepsilon > 0$. Fix n_0 such that $2^{-n_0} < \min\{\varepsilon, \frac{b-a}{4}\}$. Pick $n \ge n_0$ and consider the partition $\{0, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{2^n-1}{2^n}, 1\}$; for

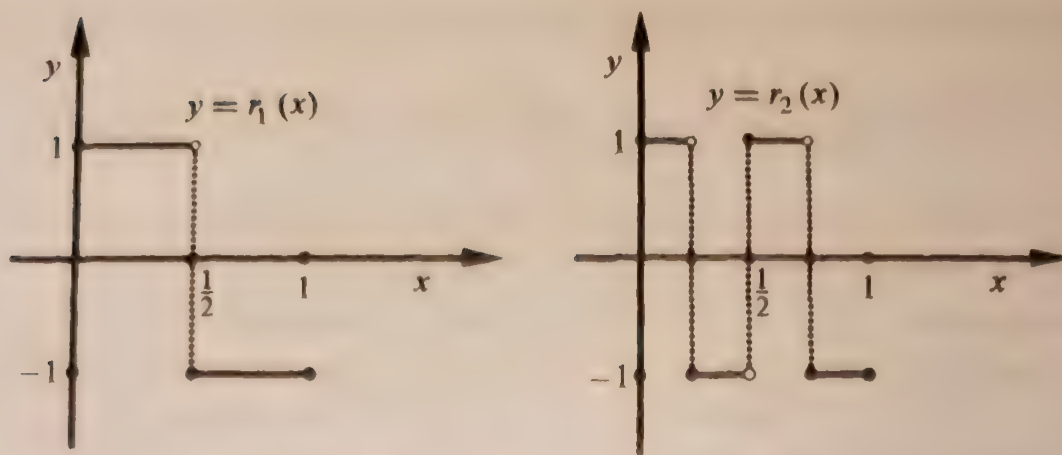

FIGURE 4.1. The Graphs of r_1 and r_2

simplicity, let $x_i = \frac{i}{2^n}$ and note that the points a and b are related to the x_i as shown in Figure 4.2.

Since for any three consecutive points x_{i-1}, x_i, x_{i+1} we have $\int_{x_{i-1}}^{x_i} r_n(x)\,dx = 0$, we see that $\int_a^b r_n(x)\,dx = \int_a^{x_k} r_n(x)\,dx + \int_c^b r_n(x)\,dx$, where $c = x_{m-1}$ or $c = x_m$; see Figure 4.2. Hence,

$$\left|\int_a^b r_n(x)\,dx\right| \le \int_a^{x_k} |r_n(x)|\,dx + \int_c^b |r_n(x)|\,dx$$
$$= (x_k - a) + (b - c) < \varepsilon + 2\varepsilon = 3\varepsilon$$

for each $n \ge n_0$. This means that $\lim \int_a^b r_n(x)\,dx = 0$, as desired.

Problem 22.23. *Let $\{\epsilon_n\}$ be a sequence of real numbers such that $0 < \epsilon_n < 1$ for each n. Also, let us say that a sequence $\{A_n\}$ of Lebesgue measurable subsets of $[0, 1]$ is consistent with the sequence $\{\epsilon_n\}$ if $\lambda(A_n) = \epsilon_n$ for each n. Establish the following properties of $\{\epsilon_n\}$:*

a. *The sequence $\{\epsilon_n\}$ converges to zero if and only if there exists a consistent sequence $\{A_n\}$ of measurable subsets of $[0, 1]$ such that $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x.*

b. *The series $\sum_{n=1}^{\infty} \epsilon_n$ converges in $\mathbf{R}$ if and only if for each consistent sequence $\{A_n\}$ of measurable subsets of $[0, 1]$ we have $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x.*

Solution. (a) If $\varepsilon_n \longrightarrow 0$, then let $A_n = (0, \varepsilon_n)$ $(n = 1, 2, \ldots)$, and note that $\lambda(A_n) = \varepsilon_n$ holds for each n and that $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for each $x \in [0, 1]$.

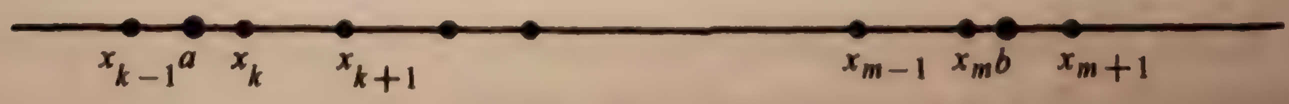

FIGURE 4.2.

For the converse, assume that there exists a consistent sequence of measurable subsets $\{A_n\}$ of $[0,1]$ satisfying $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x. For each n let $B_n = \bigcup_{k=n}^{\infty} A_k$ and note that $B_n \downarrow B = \bigcap_{k=1}^{\infty} B_k$. If $\lambda(B) > 0$ holds, then note that $\sum_{n=1}^{\infty} \chi_{A_n}(x) = \infty$ for each $x \in B$ (why?), which contradicts our hypothesis. Thus, $\lambda(B) = 0$. From the continuity of the measure (Theorem 15.4), we see that $\lambda(B_n) \downarrow 0$. In view of $A_n \subseteq B_n$, we have $0 < \varepsilon_n = \lambda(A_n) \leq \lambda(B_n)$ for each n, and so $\lim \varepsilon_n = 0$.

(b) Assume $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and that $\{A_n\}$ is a consistent sequence of measurable subsets of $[0,1]$. Then,

$$\sum_{n=1}^{\infty} \int_{[0,1]} \chi_{A_n} d\lambda = \sum_{n=1}^{\infty} \lambda(A_n) = \sum_{n=1}^{\infty} \varepsilon_n < \infty,$$

and so, by the series version of Levi's Theorem 22.9, we have $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x.

For the converse, assume that for every consistent sequence $\{A_n\}$ of measurable subsets of $[0,1]$, we have $\sum_{n=1}^{\infty} \chi_{A_n}(x) < \infty$ for almost all x. Suppose by way of contradiction that $\sum_{n=1}^{\infty} \varepsilon_n = \infty$. Using an inductive argument (how?), we see that there is a sequence $\{k_n\}$ of strictly increasing natural numbers such that $\sum_{i=k_n+1}^{k_{n+1}} \varepsilon_i > 1$ holds for each n. Next, for each n we can choose (how?) subintervals $A_{k_n+1}, A_{k_n+2}, \ldots, A_{k_{n+1}}$ of $[0,1]$ such that $\lambda(A_i) = \varepsilon_i$ for $k_n + 1 \leq i \leq k_{n+1}$ and $\bigcup_{i=k_n+1}^{k_{n+1}} A_i = [0,1]$. Now, note that the sequence of measurable sets $\{A_n\}$ is consistent with $\{\varepsilon_n\}$ and $\sum_{n=1}^{\infty} \chi_{A_n}(x) = \infty$ holds for each $x \in [0,1]$, contrary to our hypothesis. So, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ must hold.

Problem 22.24. *Let $(X, \mathcal{S}, \mu)$ be a finite measure space and let $f: X \to \mathbf{R}$ be a measurable function.*

a. *Show that if f^n is integrable for each n and that $\lim \int f^n \, d\mu$ exists in $\mathbf{R}$, then $|f(x)| \leq 1$ holds for almost all x.*
b. *If f^n is integrable for each n, then show that $\int f^n \, d\mu = c$ (a constant) for $n = 1, 2, \ldots$ if and only if $f = \chi_A$ for some measurable subset A of X.*

Solution. Keep in mind that f^n denotes the function $f^n: X \longrightarrow \mathbf{R}$ defined by $f^n(x) = [f(x)]^n$ for each $x \in X$.

(a) Assume that f^n is Lebesgue integrable for each n and that $\lim \int f^n \, d\mu$ exists in $\mathbf{R}$. Assume by way of contradiction that the measurable set

$$E = \{x \in X : |f(x)| > 1\}$$

satisfies $\mu^*(E) > 0$. From the identity $E = \bigcup_{k=1}^{\infty} E_k$, where

$$E_k = \{x \in X : |f(x)| \geq 1 + \tfrac{1}{k}\},$$

we see that there exists some $\delta > 1$ such that the measurable set

$F = \{x \in X: |f(x)| > \delta\}$ satisfies $\mu^*(F) > 0$. Now, note $f^{2n} \geq \delta^{2n}\chi_F$ holds for each n, and so from

$$\delta^{2n}\mu^*(F) = \int \delta^{2n}\chi_F \, d\mu \leq \int f^{2n} \, d\mu,$$

we infer that $\lim \int f^{2n} \, d\mu = \infty$, contradicting the existence in $\mathbb{R}$ of the limit $\lim \int f^n \, d\mu$. Hence, $|f(x)| \leq 1$ must hold for almost all x.
(b) Assume $\int f^n \, d\mu = c$ holds for each $n = 1, 2, \ldots$. By part (a), we know that $|f(x)| \leq 1$ holds for almost all $x \in X$. Now, define the sets $A = \{x \in X: f(x) = 1\}$, $B = \{x \in X: f(x) = -1\}$, and $C = \{x \in X: |f(x)| < 1\}$. Then, for each n we have

$$\begin{aligned}\int f^n \, d\mu &= \int_A f^n \, d\mu + \int_B f^n \, d\mu + \int_C f^n \, d\mu \\ &= \int_A 1 \, d\mu + \int_B (-1)^n \, d\mu + \int_C f^n \, d\mu \\ &= \mu^*(A) + (-1)^n \mu^*(B) + \int_C f^n \, d\mu = c.\end{aligned}$$

Since $f^n(x) \longrightarrow 0$ holds for each $x \in C$, it follows from the Lebesgue Dominated Convergence Theorem that $\lim \int_C f^n \, d\mu = 0$. Hence,

$$\lim_{n\to\infty} \left[\mu^*(A) + (-1)^n \mu^*(B)\right] = c.$$

Since $\lim(-1)^n$ does not exist, we infer that $\mu^*(B) = 0$, and therefore, $c = \mu^*(A) = \mu^*(A) + \int_C f^n \, d\mu$ for each n. In particular, we have $\int_C f^2 \, d\mu = 0$, and so $f(x) = 0$ must hold for almost all $x \in C$. The latter implies that $f = \chi_A$ a.e. holds.

23. THE RIEMANN INTEGRAL AS A LEBESGUE INTEGRAL

Problem 23.1. *Let $f: [a, b] \to \mathbb{R}$ be Riemann integrable. Show that f is Riemann integrable on every closed subinterval of $[a, b]$. Also, show that*

$$\int_c^d f(x)\,dx = \int_c^e f(x)\,dx + \int_e^d f(x)\,dx$$

holds for every three points c, d, and e of $[a, b]$.

Solution. Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a Riemann integrable function and let $[u,v]$ be a closed subinterval of $[a,b]$. If $f\colon [u,v] \longrightarrow \mathbb{R}$ is discontinuous at some point $x \in [u,v]$, then $f\colon [a,b] \longrightarrow \mathbb{R}$ is also discontinuous at the point x—note that, in this case, there exists a sequence $\{x_n\}$ of $[u,v]$ (and hence of $[a,b]$) such that $\{f(x_n)\}$ does not converge to $f(x)$. Thus, the set D of all points of discontinuity of $f\colon [u,v] \longrightarrow \mathbb{R}$ is a subset of the set of all points of discontinuity of $f\colon [a,b] \longrightarrow \mathbb{R}$. Since $f\colon [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable, we know that $\lambda(D) = 0$, and so (by Theorem 23.7) the function $f\colon [u,v] \longrightarrow \mathbb{R}$ is Riemann integrable.

Now, assume that $a \le c < e < d \le b$. Since $f\colon [c,e] \longrightarrow \mathbb{R}$ is Riemann integrable (and hence Lebesgue integrable), there exists a sequence of step functions $\{\phi_n\}$ over $[c,e]$ (i.e., $\phi_n(x) = 0$ holds for $x \notin [c,e]$) with $\phi_n(x) \uparrow f(x)$ for almost all $x \in [c,e]$. Similarly, there exists a sequence of step functions $\{\psi_n\}$ over $[e,d]$ such that $\psi_n(x) \uparrow f(x)$ holds for almost all $x \in [e,d]$. Then, $\{\phi_n + \psi_n\}$ is a sequence of step functions over $[c,d]$ satisfying $\phi_n(x) + \psi_n(x) \uparrow f(x)$ for almost all $x \in [c,d]$. Therefore,

$$\begin{aligned}
\int_c^d f(x)\,dx &= \int_{[c,d]} f\,d\lambda = \lim_{n\to\infty} \int_{[c,d]} (\phi_n + \psi_n)\,d\lambda \\
&= \lim_{n\to\infty} \int_{[c,d]} \phi_n\,d\lambda + \lim_{n\to\infty} \int_{[c,d]} \psi_n\,d\lambda \\
&= \int_{[c,e]} f\,d\lambda + \int_{[e,d]} f\,d\lambda \\
&= \int_c^e f(x)\,dx + \int_e^d f(x)\,dx.
\end{aligned}$$

Now, the equality $\int_c^d f(x)\,dx = \int_c^e f(x)\,dx + \int_e^d f(x)\,dx$ for arbitrary elements c, d, and e of $[a,b]$ can be obtained by considering all possible cases. We prove it for one such case and leave the rest for the reader. Assume that $a \le e < c < d \le b$. Then, by the preceding case, we have

$$\int_e^d f(x)\,dx = \int_e^c f(x)\,dx + \int_c^d f(x)\,dx = -\int_c^e f(x)\,dx + \int_c^d f(x)\,dx,$$

from which it follows that $\int_c^e f(x)\,dx + \int_e^d f(x)\,dx = \int_c^d f(x)\,dx$.

Problem 23.2. *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Then, show that*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^n f\Big(a + \frac{i(b-a)}{n}\Big).$$

Solution. The conclusion follows from Theorem 23.5 by observing that

$$\tfrac{b-a}{n}\sum_{i=1}^{n} f\left(a+i\tfrac{b-a}{n}\right) = S_f(P_n, T_n),$$

where the partition $P_n = \{x_0, x_1, \dots, x_n\}$ and $T = \{t_1, \dots, t_n\}$ satisfy $x_i = a + i\frac{b-a}{n}$ $(0 \le i \le n)$ and $t_i = x_i$ $(1 \le i \le n)$.

Problem 23.3. *Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$ such that $\{f_n\}$ converges uniformly to a function f. Show that f is Riemann integrable and that*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Solution. Choose some k such that $|f_k(x) - f_n(x)| < 1$ holds for all $n > k$ and all $x \in [a, b]$. Thus, $|f_k(x) - f(x)| \le 1$ holds for all $x \in [a, b]$. Since f_k is bounded, it is easy to see that there exists some $M > 0$ such that $|f(x)| \le M$ holds for all $x \in [a, b]$.

If $D_n \subseteq [a, b]$ denotes the set of discontinuities of f_n, then (by Theorem 23.7) $D = \bigcup_{n=1}^{\infty} D_n$ satisfies $\lambda(D) = 0$. Since each f_n is continuous on $[a, b] \setminus D$, it follows from Theorem 9.2 that f is continuous on $[a, b] \setminus D$, i.e., f is continuous almost everywhere. By Theorem 23.7, f is Riemann integrable.

For the last part, let $\epsilon > 0$. Pick some k such that $|f_n(x) - f(x)| < \epsilon$ for all $n \ge k$ and all $x \in [a, b]$. So, for $n \ge k$ we have

$$\left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| \le \int_a^b \left|f_n(x) - f(x)\right| dx < \epsilon(b-a),$$

and this shows that $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Problem 23.4. *For each n, let $f_n(x)\colon [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx^{n-1}}{1+x}$ for all $x \in [0, 1]$. Then, show that $\lim \int_0^1 f_n(x)\,dx = \frac{1}{2}$.*

Solution. Integrating by parts, we get

$$\int_0^1 f_n(x)\,dx = \frac{x^n}{1+x}\Big|_0^1 + \int_0^1 \frac{x^n}{(1+x)^2}\,dx = \frac{1}{2} + \int_0^1 \frac{x^n}{(1+x)^2}\,dx. \qquad (\star)$$

Since $0 \le \frac{x^n}{(1+x)^2} \le 1$ holds for all $x \in [0, 1]$ and $\lim \frac{x^n}{(1+x)^2} = 0$ for each x in

$[0, 1)$, the Lebesgue Dominated Convergence Theorem yields $\lim \int_0^1 \frac{x^n}{(1+x)^2}\,dx = 0$. Thus, from $(\star)$, we see that $\lim \int_0^1 f_n(x)\,dx = \frac{1}{2}$.

Problem 23.5. *Let $f\colon [a, b] \to \mathbb{R}$ be an increasing function. Show that f is Riemann integrable.*

Solution. Let $P_n = \{x_0, x_1, \ldots, x_n\}$ be the partition that subdivides $[a, b]$ into n subintervals of equal length $\frac{b-a}{n}$. Since f is increasing, note that $m_i = f(x_{i-1})$ and $M_i = f(x_i)$ hold for each $1 \le i \le n$. Next, observe that

$$\begin{aligned} 0 \le I^*(f) - I_*(f) &\le S^*(f, P_n) - S_*(f, P_n) \\ &= \sum_{i=1}^{n} \big[f(x_i) - f(x_{i-1})\big](x_i - x_{i-1}) \\ &= \big[f(b) - f(a)\big] \cdot \tfrac{b-a}{n}, \end{aligned}$$

holds for each n. Thus, $I^*(f) - I_*(f) = 0$ and so f is Riemann integrable.

An alternate proof goes as follows: According to Problem 9.8 the set of discontinuities of f is at-most countable—and hence, it has Lebesgue measure zero. Now, Theorem 23.7 guarantees that f is Riemann integrable. (See also Problem 21.8.)

Problem 23.6 (The Fundamental Theorem of Calculus). *If $f\colon [a, b] \to \mathbb{R}$ is a Riemann integrable function, then define its* **area function** $A\colon [a, b] \to \mathbb{R}$ *by $A(x) = \int_a^x f(t)\,dt$ for each $x \in [a, b]$. Show that*

a. *A is a uniformly continuous function.*
b. *If f is continuous at some point c of $[a, b]$, then A is differentiable at c and $A'(c) = f(c)$ holds.*
c. *Give an example of a Riemann integrable function f whose area function A is differentiable and satisfies $A' \neq f$.*

Solution. (a) Choose some $M > 0$ with $|f(x)| \le M$ for each x in $[a, b]$. The uniform continuity of A follows from the inequalities

$$|A(x) - A(y)| = \left|\int_x^y f(t)\,dt\right| \le \left|\int_x^y |f(t)|\,dt\right| \le M|x - y|.$$

(b) Let f be continuous at some point $c \in [a, b]$ and let $\varepsilon > 0$. Choose some $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ holds whenever $x \in [a, b]$ satisfies $|x - c| < \delta$. Then, for $x \in [a, b]$ with $0 < |x - c| < \delta$ we have $\big|f(t) - f(c)\big| < \epsilon$ for all t in

the interval with endpoints x and c, and so

$$\begin{aligned}\left|\frac{A(x)-A(c)}{x-c}-f(c)\right| &= \frac{1}{|x-c|}\left|\int_x^c f(t)\,dt - f(c)(x-c)\right| \\ &= \tfrac{1}{|x-c|}\left|\int_x^c \left[f(t)-f(c)\right]dt\right| \\ &\le \tfrac{1}{|x-c|}\cdot\varepsilon|x-c| = \varepsilon.\end{aligned}$$

This shows that $A'(c)$ exists and that $A'(c) = f(c)$ holds.

(c) We consider the function $f\colon \mathbb{R} \to \mathbb{R}$ of Problem 9.7 defined by $f(x) = 0$ if x is irrational and $f(x) = \frac{1}{n}$ if $x = \frac{m}{n}$ with $n > 0$ and with the integers m and n without having any common factors other than ± 1. It was proven in Problem 9.7 that f is continuous at every irrational and discontinuous at every rational. This implies that f restricted on an arbitrary closed interval $[c, d]$ is continuous almost everywhere and $f = 0$ a.e. From Theorems 23.6 and 23.7, we infer that f is Riemann integrable over $[c, d]$ and $\int_c^d f(x)\,dx = \int f\,d\lambda = 0$.

In particular, if $[a, b]$ is any closed interval, then $A(x) = \int_a^x f(t)\,dt = 0$ for each $x \in [a, b]$. Thus, $A'(x) = 0$ for each $x \in [a, b]$, and consequently $A'(x) \neq f(x)$ at each rational number x in $[a, b]$.

Problem 23.7 (Arzelà). *Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a, b]$ such that $\lim f_n(x) = f(x)$ holds for each $x \in [a, b]$ and f is Riemann integrable. Also, assume that there exists a constant M such that $|f_n(x)| \le M$ holds for all $x \in [a, b]$ and all n. Show that*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Solution. Using Theorem 23.6 and the Lebesgue Dominated Convergence Theorem, we see that

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,d\lambda = \lim_{n\to\infty}\int_{[a,b]} f_n\,d\lambda = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$$

Problem 23.8. *Determine the lower and upper Riemann integrals for the function $f\colon [0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = 0$ if x is a rational number and $f(x) = 1$ if x is an irrational number.*

Solution. Let P be a partition of $[0, 1]$. Since each interval contains rational and irrational numbers, we have $m_i = 0$ and $M_i = 1$ for all i. Thus, $S^*(f, P) = 1$ and $S_*(f, P) = 0$ for all partitions P. Therefore, $I^*(f) = 1$ and $I_*(f) = 0$.

Problem 23.9. *Let C be the Cantor set (see Example 6.15). Show that χ_C is Riemann integrable over $[0, 1]$, and that $\int_0^1 \chi_C\, dx = 0$.*

Solution. Note that χ_C is continuous at every point of $[0, 1] \setminus C$ and discontinuous at every point of C. Since $\lambda(C) = 0$, it follows from Theorem 23.7 that χ_C is Riemann integrable over $[0, 1]$. Since $\chi_C = 0$ a.e. holds, we see that

$$\int_0^1 \chi_C(x)\, dx = \int_{[0,1]} \chi_C\, d\lambda = 0.$$

Problem 23.10. *Let $0 < \epsilon < 1$, and consider the ϵ-Cantor set C_ϵ of $[0, 1]$. Show that χ_{C_ϵ} is not Riemann integrable over $[0, 1]$. Also, determine $I_*(\chi_{C_\epsilon})$ and $I^*(\chi_{C_\epsilon})$.*

Solution. Consider the ε-Cantor set for some $0 < \varepsilon < 1$. Since C_ε is nowhere dense in $[0, 1]$, it is easy to see that χ_{C_ε} is discontinuous at every point of C_ε and continuous at every point of $[0, 1] \setminus C_\varepsilon$. Since $\lambda(C_\varepsilon) = \varepsilon > 0$, it follows from Theorem 23.7 that χ_{C_ε} is not Riemann integrable.

Now, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0, 1]$. Since C_ε is nowhere dense, it follows that $m_i = 0$ for each $1 \le i \le n$. Thus, $S_*(\chi_{C_\varepsilon}, P) = 0$ for each partition P, and so $I_*(\chi_{C_\varepsilon}) = 0$. Clearly, $M_i = 1$ if $[x_{i-1}, x_i] \cap C_\varepsilon \neq \emptyset$, and $M_i = 0$ if $[x_{i-1}, x_i] \cap C_\varepsilon = \emptyset$. Since $C_\varepsilon = \bigcup_{i=1}^n [x_{i-1}, x_i] \cap C_\varepsilon$, it follows that

$$\varepsilon = \lambda(C_\varepsilon) \le \sum_{i=1}^n \lambda\big([x_{i-1}, x_i] \cap C_\varepsilon\big) \le \sum_{i=1}^n M_i(x_i - x_{i-1}) = S^*(\chi_{C_\varepsilon}, P),$$

and so $I^*(\chi_{C_\varepsilon}) \ge \varepsilon$.

On the other hand, if $0 < \delta < 1 - \varepsilon$, then there exist pairwise disjoint open subintervals $(a_1, b_1), \ldots, (a_n, b_n)$ such that

$$[a_i, b_i] \subseteq [0, 1] \setminus C_\varepsilon \ (1 \le i \le n) \quad \text{and} \quad -\sum_{i=1}^n (b_i - a_i) > 1 - \varepsilon - \delta.$$

The endpoints of all these subintervals together with 0 and 1 form a partition P

of $[0, 1]$ such that

$$\varepsilon \leq I^*(\chi_{C_\varepsilon}) \leq S^*(\chi_{C_\varepsilon}, P) \leq 1 - (1 - \varepsilon - \delta) = \varepsilon + \delta.$$

Since $0 < \delta < 1 - \varepsilon$ is arbitrary, it easily follows that $I^*(\chi_{C_\varepsilon}) = \varepsilon$.

Problem 23.11. *Give a proof of the Riemann integrability of a continuous function based upon its uniform continuity (Theorem 7.7).*

Solution. Let $\varepsilon > 0$. Since (by Theorem 7.7) f is uniformly continuous, there is some $\delta > 0$ such that $x, y \in [a, b]$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Let P be a partition of $[a, b]$ with mesh $|P| < \delta$. Then, $M_i - m_i < \varepsilon$ holds for each $1 \leq i \leq n$ (why?), and so

$$0 \leq I^*(f) - I_*(f) \leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon(b - a).$$

Since $\varepsilon > 0$ is arbitrary, we see that $I^*(f) = I_*(f)$ holds, and therefore f is Riemann integrable.

Problem 23.12. *Establish the following change of variable formula for the Riemann integral of continuous functions: If $[a, b] \xrightarrow{g} [c, d] \xrightarrow{f} \mathbb{R}$ are continuous functions with g continuously differentiable (i.e., g has a continuous derivative), then*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Solution. We shall apply the Fundamental Theorem of Calculus in connection with the Chain Rule. We consider the two functions $F, G\colon [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_{g(a)}^{g(x)} f(u)\,du \quad \text{and} \quad G(x) = \int_a^x f\big(g(x)\big)g'(x)\,dx$$

for all $x \in [a, b]$. Next, we shall compute the derivatives of F and G separately. For the derivative of F we use the Fundamental Theorem of Calculus and the Chain Rule to get $F'(x) = f\big(g(x)\big)g'(x)$ for each $x \in [a, b]$. For the derivative of G, the Fundamental Theorem of Calculus yields $G'(x) = f\big(g(x)\big)g'(x)$ for each $x \in [a, b]$. So, $F'(x) = G'(x)$ for all $x \in [a, b]$.

The latter implies that there exists a constant c such that $F(x) = G(x) + c$ for all $x \in [a, b]$. Letting $x = a$ and taking into account that $F(a) = G(a) = 0$,

we get $c = 0$. Thus, $F(x) = G(x)$ for all $x \in [a, b]$. Finally, letting $x = b$, we obtain

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du,$$

as desired.

Problem 23.13. *Let $f: [0, \infty) \to \mathbf{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \delta$. Show that $\lim_{n\to\infty} \int_0^a f(nx)\,dx = a\delta$ for each $a > 0$.*

Solution. Fix $a > 0$ and then define the sequence of continuous functions $\{f_n\}$ by $f_n(x) = f(nx)$. Clearly, $\lim_{n\to\infty} f_n(x) = \delta$ holds for all $x \in (0, a]$. We claim that the sequence of functions $\{f_n\}$ is uniformly bounded on the interval $[0, a]$. Indeed, since $\lim_{x\to\infty} f(x) = \delta$ holds, there exists a number $M > 0$ such that $|f(x)| < |\delta| + 1$ whenever $x > M$. Also, since f is a continuous function, it is bounded on the interval $[0, M]$. Thus, there exists a constant C such that $|f(x)| \le C$ holds for all x, and hence $|f_n(x)| = |f(nx)| \le C$ holds for all x. Now, an application of the Lebesgue Dominated Convergence yields

$$\lim_{n\to\infty} \int_0^a f(nx)\,dx = \lim_{n\to\infty} \int_0^a f_n(x)\,dx = \int_0^a \delta\,dx = a\delta.$$

Problem 23.14. *Let $f: [0, \infty) \to \mathbf{R}$ be a real-valued continuous function such that $f(x+1) = f(x)$ for all $x \ge 0$. If $g: [0, 1] \to \mathbf{R}$ is an arbitrary continuous function, then show that*

$$\lim_{n\to\infty} \int_0^1 g(x)f(nx)\,dx = \Big(\int_0^1 g(x)\,dx\Big)\cdot\Big(\int_0^1 f(x)\,dx\Big).$$

Solution. Let the functions f and g satisfy the hypotheses of the problem. Observe first that an easy inductive argument establishes that $f(x+k) = f(x)$ holds for all $x \ge 0$ and all non-negative integers k.

The change of variable $u = nx$ yields

$$\begin{aligned}\int_0^1 g(x)f(nx)\,dx &= \tfrac{1}{n}\int_0^n g(\tfrac{u}{n})f(u)\,du\\ &= \tfrac{1}{n}\sum_{i=1}^n \int_{i-1}^i g(\tfrac{u}{n})f(u)\,du.\end{aligned}$$

Letting $t = u - i + 1$, we get

$$\int_{i-1}^{i} g(\tfrac{u}{n}) f(u)\,du = \int_0^1 g(\tfrac{t+i-1}{n}) f(t+i-1)\,dt = \int_0^1 g(\tfrac{t+i-1}{n}) f(t)\,dt.$$

Consequently,

$$\int_0^1 g(x) f(nx)\,dx = \int_0^1 \Big[\sum_{i=1}^{n} \tfrac{1}{n} g(\tfrac{t+i-1}{n})\Big] f(t)\,dt = \int_0^1 h_n(t)\,dt, \qquad (\star)$$

where $h_n(t) = \big[\sum_{i=1}^{n} \frac{1}{n} g(\frac{t+i-1}{n})\big] f(t)$. Clearly, h_n is a continuous function defined on $[0, 1]$. In addition, note that if $|g(x)| \le K$ and $|f(x)| \le K$ hold for each $x \in [0, 1]$, then

$$|h_n(t)| \le K^2 \quad \text{for all} \quad t \in [0, 1],$$

i.e., the sequence $\{h_n\}$ is uniformly bounded on $[0, 1]$. Next, note that $0 \le t \le 1$ implies $\frac{i-1}{n} \le \frac{t+i-1}{n} \le \frac{i}{n}$. Thus, if m_i^n and M_i^n denote the minimum and maximum values of g, respectively, on the closed interval $[\frac{i-1}{n}, \frac{i}{n}]$, then

$$m_i^n \le g(\tfrac{t+i-1}{n}) \le M_i^n$$

holds for each $0 \le t \le 1$. Next, put

$$R_n = \sum_{i=1}^{n} \tfrac{1}{n} m_i^n \quad \text{and} \quad S_n = \sum_{i=1}^{n} \tfrac{1}{n} M_i^n,$$

and note that R_n and S_n are two Riemann sums—the smallest and largest ones, respectively—for the function g corresponding to the partition $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$. Hence, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} S_n = \int_0^1 g(x)\,dx$. From

$$\begin{aligned}
|h_n(t) - R_n \cdot f(t)| &= \Big|\Big[\sum_{i=1}^{n} \tfrac{1}{n} g(\tfrac{t+i-1}{n})\Big] f(t) - R_n \cdot f(t)\Big| \\
&= \Big(\Big[\sum_{i=1}^{n} \tfrac{1}{n} g(\tfrac{t+i-1}{n})\Big] - R_n\Big) \cdot |f(t)| \\
&\le (S_n - R_n)|f(t)|,
\end{aligned}$$

we see that $\lim_{n\to\infty} h_n(t) = f(t)\int_0^1 g(x)\,dx$—in fact, the sequence $\{h_n\}$ converges uniformly (why?).

Now, use $(\star)$ and the Lebesgue Dominated Convergence Theorem to obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 g(x)f(nx)\,dx &= \lim_{n\to\infty}\int_0^1 h_n(t)\,dt\\ &= \int_0^1 \big[\lim_{n\to\infty} h_n(t)\big]\,dt\\ &= \int_0^1 \Big[f(t)\int_0^1 g(x)\,dx\Big]\,dt\\ &= \Big(\int_0^1 f(t)\,dt\Big)\cdot\Big(\int_0^1 g(x)\,dx\Big).\end{aligned}$$

Problem 23.15. *Let $f\colon [0,1]\to[0,\infty)$ be Riemann integrable on every closed subinterval of $(0,1]$. Show that f is Lebesgue integrable over $[0,1]$ if and only if $\lim_{\epsilon\downarrow 0}\int_\epsilon^1 f(x)\,dx$ exists in $\mathbf{R}$. Also, show that if this is the case, then we have $\int f\,d\lambda = \lim_{\epsilon\downarrow 0}\int_\epsilon^1 f(x)\,dx$.*

Solution. Assume that f is Lebesgue integrable. Let $\{\varepsilon_n\}$ be an arbitrary sequence of $(0,1]$ with $\varepsilon_n \downarrow 0$. For each n, we consider the upper function $g_n = f\chi_{[\varepsilon_n,1]}$. Then, $g_n \uparrow f$ a.e. holds and so, by Theorem 21.6, we have

$$\int f\,d\lambda = \lim_{n\to\infty}\int g_n\,d\lambda = \lim_{n\to\infty}\int_{\varepsilon_n}^1 f(x)\,dx.$$

This shows that $\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 f(x)\,dx$ exists and that

$$\lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 f(x)\,dx = \int f\,d\lambda.$$

Conversely, assume that the limit exists. Let $\varepsilon_n = \frac{1}{n}$ and consider the sequence of upper functions $\{g_n\}$ as previously (i.e., let $g_n = f\chi_{[\varepsilon_n,1]}$). Then, $g_n \uparrow f$ a.e. and

$$\lim_{n\to\infty}\int g_n\,d\lambda = \lim_{n\to\infty}\int_{\varepsilon_n}^1 f(x)\,dx = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 f(x)\,dx < \infty.$$

By Theorem 21.6, f is an upper function, and hence, Lebesgue integrable.

Problem 23.16. *As an application of the preceding problem, show that the function $f: [0, 1] \to \mathbb{R}$ defined by $f(x) = x^p$ if $x \in (0, 1]$ and $f(0) = 0$ is Lebesgue integrable if and only if $p > -1$. Also, show that if f is Lebesgue integrable, then*

$$\int f \, d\lambda = \frac{1}{1+p}.$$

Solution. If $0 < \varepsilon < 1$, then note that $\int_\varepsilon^1 x^p \, dx = \frac{1-\varepsilon^{p+1}}{p+1}$ for $p \neq -1$ and $\int_\varepsilon^1 x^{-1} \, dx = -\ln \varepsilon$. Thus, $\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 x^p \, dx$ exists if and only if $p > -1$, and, in this case, the limit is $\frac{1}{p+1}$. The conclusion now follows immediately from the preceding problem.

Problem 23.17. *Let $f: [0, 1] \to \mathbb{R}$ be a function and define $g: [0, 1] \to \mathbb{R}$ by $g(x) = e^{f(x)}$.*

a. *Show that if f is measurable (or Borel measurable), then so is g.*
b. *If f is Lebesgue integrable, is then g necessarily Lebesgue integrable?*
c. *Give an example of an essentially unbounded function f which is continuous on $(0, 1]$ such that f^n is Lebesgue integrable for each $n = 1, 2, \ldots$. (A function f is "essentially unbounded," if for each positive real number $M > 0$ the set $\{x \in [0, 1]: |f(x)| > M\}$ has positive measure.)*

Solution. (a) Let $h(x) = e^x$ and note that $g = h \circ f$. The conclusion follows from the identity $(h \circ f)^{-1}(V) = f^{-1}\big(g^{-1}(V)\big)$ and the fact that h is a continuous function.

(b) The measurable function g need not be necessarily Lebesgue integrable. Here is an example. Consider the function $f: [0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{\sqrt{x}}$; at $x = 0$ we let $f(0) = 0$. If $0 < \varepsilon < 1$, then the change of variable $t = \sqrt{x}$ yields

$$\int_\varepsilon^1 f(x) \, dx = \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = 2 \int_{\sqrt{\varepsilon}}^1 dt = 2(1 - \sqrt{\varepsilon}).$$

Therefore, from Problem 23.16, we see that f is Lebesgue integrable and $\int f \, d\lambda = 2$. On the other hand, for each $0 < \epsilon < 1$ the change of variable $u = \frac{1}{\sqrt{x}}$ yields

$$\int_\varepsilon^1 g(x) \, d\lambda(x) = \int_\varepsilon^1 e^{\frac{1}{\sqrt{x}}} \, dx = 2 \int_1^{\frac{1}{\sqrt{\varepsilon}}} \frac{e^u}{u^3} \, du \geq 2 \int_1^{\frac{1}{\sqrt{\varepsilon}}} e^u \, du = 2\big(e^{\frac{1}{\sqrt{\varepsilon}}} - 1\big).$$

This implies $\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 e^{\frac{1}{\sqrt{x}}} \, dx = \infty$, and so by Problem 23.15 the function g is not Lebesgue integrable over $[0, 1]$.

(c) The function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=\ln x$ for $0<x\le 1$ and $f(0)=0$ satisfies $\int_0^1 f^n(x)\,d\lambda(x)=(-1)^n n!$ for each $n=1,2,\ldots$.

Problem 23.18. *Let $f:[0,1]\to\mathbb{R}$ be Lebesgue integrable (with respect to the Lebesgue measure). Assume that f is differentiable at $x=0$ and $f(0)=0$. Show that the function $g:[0,1]\to\mathbb{R}$ defined by $g(x)=x^{-\frac{3}{2}}f(x)$ for $x\in(0,1]$ and $g(0)=0$ is Lebesgue integrable.*

Solution. Start by observing that (by Problem 23.16) the function $h(x)=x^{-\frac{1}{2}}$ for $x\in(0,1]$ is Lebesgue integrable over $[0,1]$. Since $f(0)=0$ and $f'(0)$ exists, there exist $0<\delta<1$ and $M>0$ such that $|f(x)|\le Mx$ for all $0\le x\le\delta$. Since for $\delta\le x\le 1$ we have $x^{-\frac{3}{2}}\le\delta^{-\frac{3}{2}}$, we can assume that $M>\delta^{-\frac{3}{2}}$. Now, note that for $0<x\le 1$, we have

$$\begin{aligned}|g(x)| = \left|x^{-\frac{3}{2}}f(x)\right| &\le M\begin{cases} x^{-\frac{1}{2}} & \text{if } 0<x\le\delta\\ |f(x)| & \text{if } \delta<x\le 1\end{cases}\\ &\le M\big(h+|f|\big)(x).\end{aligned}$$

Since $h+|f|$ is integrable and (obviously) g is measurable, Theorem 22.6 guarantees that g is also Lebesgue integrable.

Problem 23.19. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a continuous function. Show that the Riemann integral of f can be computed with two iterated integrations. That is, show that*

$$\int_a^b\int_c^d f(x,y)\,dx\,dy=\int_a^b\Big[\int_c^d f(x,y)\,dy\Big]dx=\int_c^d\Big[\int_a^b f(x,y)\,dx\Big]dy.$$

Generalize this to a continuous function of n variables.

Solution. Note first that the functions

$$x\longmapsto\int_c^d f(x,y)\,dy \quad\text{and}\quad y\longmapsto\int_a^b f(x,y)\,dx$$

are both continuous—and so both iterated integrals are well defined. Indeed, since the function f is uniformly continuous, given $\varepsilon>0$ there exists some $\delta>0$ such that $|x_1-x_2|<\delta$ and $|y_1-y_2|<\delta$ imply $|f(x_1,y_1)-f(x_2,y_2)|<\varepsilon$.

Thus, if $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$ both hold, then

$$\left|\int_c^d f(x_1, y)\,dy - \int_c^d f(x_2, y)\,dy\right| < \varepsilon(c - d)$$

and

$$\left|\int_a^b f(x, y_1)\,dx - \int_a^b f(x, y_2)\,dx\right| < \varepsilon(b - a).$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and $Q = \{y_0, y_1, \ldots, y_k\}$ be a partition of $[c, d]$. Put $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and then define

$$m_{ij} = \inf\{f(x, y) \colon (x, y) \in R_{ij}\} \text{ and } M_{ij} = \sup\{f(x, y) \colon (x, y) \in R_{ij}\}.$$

From the inequality $m_{ij} \le f(x, y) \le M_{ij}$ for $(x, y) \in R_{ij}$, it follows that

$$m_{ij}(y_j - y_{j-1}) \le \int_{y_{j-1}}^{y_j} f(x, y)\,dy \le M_{ij}(y_j - y_{j-1})$$

for all $x_{i-1} \le x \le x_i$, and so

$$\begin{aligned} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) &\le \int_{x_{i-1}}^{x_i} \left(\int_{y_{j-1}}^{y_j} f(x, y)\,dy\right) dx \\ &\le M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}). \end{aligned}$$

Consequently, we have

$$\begin{aligned} S_*(f, P \times Q) &= \sum_{i=1}^{n}\sum_{j=1}^{k} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{k} \int_{x_{i-1}}^{x_i} \left(\int_{y_{j-1}}^{y_j} f(x, y)\,dy\right) dx \\ &= \int_a^b \left(\int_c^d f(x, y)\,dy\right) dx \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{k} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \\ &= S^*(f, P \times Q). \end{aligned}$$

Since P and Q are arbitrary and f is Riemann integrable, it follows that

$$\int_a^b \int_c^d f(x, y)\,dx\,dy = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx.$$

The other equality can be proven in a similar manner.

Problem 23.20. *Assume that $f: [a, b] \to \mathbf{R}$ and $g: [a, b] \to \mathbf{R}$ are two continuous functions such that $f(x) \le g(x)$ for each $x \in [a, b]$. Let*

$$A = \{(x, y) \in \mathbf{R}^2 \colon x \in [a, b] \text{ and } f(x) \le y \le g(x)\}.$$

a. *Show that A is a closed set—and hence, a measurable subset of $\mathbf{R}^2$.*
b. *If $h: A \to \mathbf{R}$ is a continuous function, then show that h is Lebesgue integrable over A and that*

$$\int_A h\,d\lambda = \int_a^b \left[\int_{f(x)}^{g(x)} h(x, y)\,dy \right] dx.$$

Solution. (a) Let $\{(x_n, y_n)\}$ be a sequence of A such that $x_n \to x$ and $y_n \to y$. From the inequality $f(x_n) \le y_n \le g(x_n)$ and the continuity of f and g, it follows that $f(x) \le y \le g(x)$, i.e., $(x, y) \in A$. Thus, A is a closed set.
(b) Let $c < \inf\{f(x)\colon x \in [a, b]\}$ and $d > \sup\{g(x)\colon x \in [a, b]\}$. Consequently, $A \subseteq [a, b] \times [c, d] = E$. Extend h to E by $h(x, y) = 0$ if $(x, y) \notin A$, and note that the set of all discontinuities of h is a subset of

$$D = \big\{(x, y) \in \mathbf{R}^2 \colon a \le x \le b \text{ and } y = f(x) \text{ or } y = g(x)\big\}.$$

By Problem 18.17, $\lambda(D) = 0$, and so h is Riemann integrable on E (and hence, Lebesgue integrable). Now, by modifying the arguments of Problem 23.19, we easily see that

$$\begin{aligned} \int_A h\,d\lambda &= \int_a^b \int_c^d h(x, y)\,dx\,dy = \int_a^b \left(\int_c^d h(x, y)\,dy \right) dx \\ &= \int_a^b \left(\int_{f(x)}^{g(x)} h(x, y)\,dy \right) dx. \end{aligned}$$

Problem 23.21. *Let $f: [a, b] \to \mathbf{R}$ be a differentiable function—with one-sided derivatives at the end points. If the derivative f' is uniformly bounded on $[a, b]$,*

then show that f' is Lebesgue integrable and that

$$\int_{[a,b]} f'\,d\lambda = f(b) - f(a).$$

Solution. Let $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ be a differentiable function such that for some $M > 0$ we have $|f'(x)| \le M$ for all $x \in [a, b]$. By letting $f(x) = f(a) + f'(a)(x - a)$ for $x < a$ and $f(x) = f(b) + f'(b)(x - b)$ for $x > b$, we can assume that f is defined (and is differentiable) on $\mathbf{R}$.

Next, consider the sequence of differentiable functions $\{f_n\}$ defined by

$$f_n(x) = n\big[f(x + \tfrac{1}{n}) - f(x)\big] = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}, \quad x \in \mathbf{R},$$

and note that $f_n(x) \longrightarrow f'(x)$ holds for each $x \in \mathbf{R}$. Also, by the Mean Value Theorem, it is easy to see that $|f_n(x)| \le M$ holds for each x. Consequently, by the Lebesgue Dominated Convergence Theorem, f' is Lebesgue integrable over $[a, b]$ and

$$\int_{[a,b]} f'\,d\lambda = \lim_{n\to\infty} \int_a^b f_n(x)\,dx. \qquad (\star)$$

Now, using the change of variable $u = x + \frac{1}{n}$, we see that

$$\begin{aligned}
\int_a^b f_n(x)\,dx &= n\Big[\int_a^b f(x + \tfrac{1}{n})\,dx - \int_a^b f(x)\,dx\Big] \\
&= n\Big[\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(u)\,du - \int_a^b f(x)\,dx\Big] \\
&= n\Big[\int_b^{b+\frac{1}{n}} f(x)\,dx - \int_a^{a+\frac{1}{n}} f(x)\,dx\Big] \\
&= \frac{\int_b^{b+\frac{1}{n}} f(x)\,dx}{\frac{1}{n}} - \frac{\int_a^{a+\frac{1}{n}} f(x)\,dx}{\frac{1}{n}} \longrightarrow f(b) - f(a),
\end{aligned}$$

where the last limit is justified by virtue of the Fundamental Theorem of Calculus. A glance at $(\star)$ guarantees that $\int_{[a,b]} f'\,d\lambda = f(b) - f(a)$, and we are finished.

Problem 23.22. *Let $f, g: [a, b] \to \mathbf{R}$ be two Lebesgue integrable functions satisfying*

$$\int_a^x f(t)\,d\lambda(t) \leq \int_a^x g(t)\,d\lambda(t)$$

for all $x \in [a, b]$. If $\phi: [a, b] \to \mathbf{R}$ is a non-negative decreasing function, then show that the functions ϕf and ϕg are both Lebesgue integrable over $[a, b]$ and that they satisfy

$$\int_a^x \phi(t) f(t)\,d\lambda(t) \leq \int_a^x \phi(t) g(t)\,d\lambda(t)$$

for all $x \in [a, b]$.

Solution. Since ϕ is decreasing there exists some $M > 0$ satisfying $|\phi(t)| \leq M$ for each $t \in [a, b]$. Since f and g are Lebesgue integrable, it follows from the inequalities $|\phi(t)f(t)| \leq M|f(t)|$ and $|\phi(t)g(t)| \leq M|g(t)|$ for each $t \in [a, b]$ that ϕf and ϕg are both Lebesgue integrable functions over $[a, b]$.

To obtain the desired inequality, fix $x \in [a, b]$. Assume first that ϕ is a non-negative decreasing function of the form $\phi = \sum_{i=1}^k c_i \chi_{[a_{i-1}, a_i)}$, where $\{a = a_0 < a_1 < \cdots < a_k = b\}$ is a partition of $[a, b]$. Since f is decreasing, we know that $c_1 \geq c_2 \geq \cdots \geq c_k \geq 0$. Clearly,

$$\begin{aligned} \phi &= (c_1 - c_2)\chi_{[a,a_1)} + (c_2 - c_3)\chi_{[a,a_2)} + \cdots + (c_{k-1} - c_k)\chi_{[a,a_{k-1})} + c_k \chi_{[a,b)} \\ &= \sum_{i=1}^k \gamma_i \chi_{[a,a_i)}, \end{aligned}$$

with $\gamma_i \geq 0$ for each i. Pick $1 \leq m \leq k$ such that $a_{m-1} \leq x < a_m$, and note that

$$\begin{aligned} \int_a^x \phi(t) f(t)\,d\lambda(t) &= \sum_{i=1}^{m-1} \gamma_i \int_a^{a_i} f(t)\,d\lambda(t) + \gamma_m \int_a^x f(t)\,d\lambda(t) \\ &\leq \sum_{i=1}^{m-1} \gamma_i \int_a^{a_i} g(t)\,d\lambda(t) + \gamma_m \int_a^x g(t)\,d\lambda(t) \\ &= \int_a^x \phi(t) g(t)\,d\lambda(t). \end{aligned}$$

Now, we consider the general case. Fix $x \in [a, b)$. As in the solution of Problem 21.8, we see that there exists a sequence $\{\phi_n\}$ of non-negative decreasing

step functions (as above) satisfying $\phi_n(t) \uparrow \phi(t)$ for almost all $t \in [a, b]$. Since $|\phi_n(t)f(t)| \le M|f(t)|$, $|\phi_n(t)g(t)| \le M|g(t)|$, $\phi_n(t)f(t) \to \phi(t)f(t)$, and since $\phi_n(t)g(t) \to \phi(t)g(t)$ for almost all $t \in [a, b]$, it follows from the inequality

$$\int_a^x \phi_n(t)f(t)\,d\lambda(t) \le \int_a^x \phi_n(t)g(t)\,d\lambda(t)$$

and the Lebesgue Dominated Convergence Theorem that

$$\begin{aligned}\int_a^x \phi(t)f(t)\,d\lambda(t) &= \lim_{n\to\infty}\int_a^x \phi_n(t)f(t)\,d\lambda(t)\\ &\le \lim_{n\to\infty}\int_a^x \phi_n(t)g(t)\,d\lambda(t) = \int_a^x \phi(t)g(t)\,d\lambda(t).\end{aligned}$$

24. APPLICATIONS OF THE LEBESGUE INTEGRAL

Problem 24.1. *Show that*

$$\int_0^\infty x^{2n}e^{-x^2}\,dx = \frac{(2n)!}{2^{2n}n!}\cdot\frac{\sqrt{\pi}}{2}$$

holds for $n = 0, 1, 2, \ldots$.

Solution. We shall establish the formula by induction on n. For $n = 0$ the formula is true by virtue of Theorem 24.6. If the formula is true for some $n \ge 0$, then an integration by parts yields

$$\begin{aligned}\int_0^r x^{2(n+1)}e^{-x^2}\,dx &= -\tfrac{1}{2}\int_0^r x^{2n+1}\,d\left(e^{-x^2}\right)\\ &= -\tfrac{1}{2}r^{2n+1}e^{-r^2} + \tfrac{2n+1}{2}\int_0^r x^{2n}e^{-x^2}\,dx\end{aligned}$$

for each $r > 0$. This implies

$$\begin{aligned}\int_0^\infty x^{2(n+1)}e^{-x^2}\,dx &= \lim_{r\to\infty}\int_0^r x^{2(n+1)}e^{-x^2}\,dx = \tfrac{2n+1}{2}\int_0^\infty x^{2n}e^{-x^2}\,dx\\ &= \tfrac{(2n+1)(2n+2)}{2^2(n+1)}\cdot\tfrac{(2n)!}{2^{2n}n!}\cdot\tfrac{\sqrt{\pi}}{2}\\ &= \tfrac{[2(n+1)]!}{2^{2(n+1)}(n+1)!}\cdot\tfrac{\sqrt{\pi}}{2}.\end{aligned}$$

Problem 24.2. *Show that* $\int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$ *for each* $t > 0$.

Solution. Let $u = x\sqrt{t}$. Then, $\int_0^r e^{-tx^2}\,dx = \frac{1}{\sqrt{t}}\int_0^{r\sqrt{t}} e^{-u^2}\,du$ holds for each $r > 0$. Therefore,

$$\begin{aligned}\int_0^\infty e^{-tx^2}\,dx &= \lim_{r\to\infty}\int_0^r e^{-tx^2}\,dx = \tfrac{1}{\sqrt{t}}\lim_{r\to\infty}\int_0^{r\sqrt{t}} e^{-u^2}\,du \\ &= \tfrac{1}{\sqrt{t}}\cdot\tfrac{\sqrt{\pi}}{2} = \tfrac{1}{2}\sqrt{\tfrac{\pi}{t}}.\end{aligned}$$

Problem 24.3. *Show that* $f(x) = \frac{\ln x}{x^2}$ *is Lebesgue integrable over* $[1,\infty)$ *and that* $\int f\,d\lambda = 1$.

Solution. Since $\frac{\ln x}{x^2} \geq 0$ holds for each $x \geq 1$, it suffices (in view of Theorem 24.3) to show that $\int_1^\infty \frac{\ln x}{x^2}\,dx$ exists.

If $r > 1$, then an integration by parts yields

$$\int_1^r \tfrac{\ln x}{x^2}\,dx = -\int_1^r \ln x\, d\big(\tfrac{1}{x}\big) = -\tfrac{\ln x}{x}\Big|_1^r + \int_1^r \tfrac{1}{x^2}\,dx = 1 - \tfrac{1}{r} - \tfrac{\ln r}{r}.$$

Therefore,

$$\int f\,d\lambda = \int_1^\infty \tfrac{\ln x}{x^2}\,dx = \lim_{r\to\infty}\int_1^r \tfrac{\ln x}{x^2}\,dx = \lim_{r\to\infty}\big(1 - \tfrac{1}{r} - \tfrac{\ln r}{r}\big) = 1.$$

Problem 24.4. *Show that*

$$\lim_{n\to\infty}\int_0^n \Big(1+\frac{x}{n}\Big)^n e^{-2x}\,dx = 1.$$

Solution. Note that

$$\int_0^\infty e^{-x}\,dx = \lim_{r\to\infty}\int_0^r e^{-x}\,dx = \lim_{r\to\infty}(1-e^{-r}) = 1.$$

Therefore, the function e^{-x} is Lebesgue integrable over $[0,\infty)$. Now, let $g_n(x) = \big(1+\frac{x}{n}\big)^n e^{-2x}\chi_{[0,n]}(x)$, and note that each g_n is Lebesgue integrable over $[0,\infty)$.

From elementary calculus, we know that $(1+\frac{x}{n})^n \uparrow e^x$ for each $x \geq 0$, and so $g_n(x) \uparrow e^{-x}$ holds for each $x \geq 0$. Thus,

$$\lim_{n\to\infty} \int_0^n (1+\tfrac{x}{n})^n e^{-x}\,dx = \lim_{n\to\infty} \int g_n\,d\lambda = \int_0^\infty e^{-x}\,dx = 1.$$

Problem 24.5. *Let $f\colon[0,\infty) \to (0,\infty)$ be a continuous, decreasing, and Lebesgue integrable function. Show that*

$$\lim_{x\to\infty} \frac{1}{f(x)} \int_x^\infty f(s)\,ds = 0 \quad \textit{if and only if} \quad \lim_{x\to\infty} \frac{f(x+t)}{f(x)} = 0 \ \textit{for each}\ t > 0.$$

Solution. Assume that $\lim_{x\to\infty} \frac{1}{f(x)} \int_x^\infty f(s)\,ds = 0$ and let $t > 0$ be fixed. Since f is decreasing, we see that $f(x+t) \leq f(s)$ for all $x \leq s \leq x+t$, and so

$$tf(x+t) = \int_x^{x+t} f(x+t)\,ds \leq \int_x^{x+t} f(s)\,ds.$$

Consequently, we have

$$0 < \frac{f(x+t)}{f(x)} \leq \frac{1}{t} \cdot \frac{\int_x^{x+t} f(s)\,ds}{f(x)} \leq \frac{1}{t} \cdot \frac{\int_x^\infty f(s)\,ds}{f(x)},$$

from which it follows that $\lim_{x\to\infty} \frac{f(x+t)}{f(x)} = 0$. For the converse, assume that $\lim_{x\to\infty} \frac{f(x+t)}{f(x)} = 0$ holds for each fixed t, and, for simplicity, let us write $F(x) = \frac{1}{f(x)} \int_x^\infty f(s)\,ds$ for each $x \in [0,\infty)$. Fix $\varepsilon > 0$ and then choose some $0 < \delta < 1$ such that $\frac{\delta}{1-\delta} < \varepsilon$. (Since $\lim_{\delta\to 0^+} \frac{\delta}{1-\delta} = 0$ such a δ always exists.) From $\lim_{x\to\infty} \frac{f(x+\delta)}{f(x)} = 0$, we infer that there exists some $M > 0$ such that $\frac{f(x+\delta)}{f(x)} < \delta$ holds for all $x > M$. That is, $f(x+\delta) \leq \delta f(x)$ holds for each $x > M$. Now, note that for $x > M$, we have

$$\begin{aligned}
F(x) &= \frac{1}{f(x)} \int_x^{x+\delta} f(s)\,ds + \frac{1}{f(x)} \int_{x+\delta}^\infty f(s)\,ds \\
&= \frac{1}{f(x)} \int_x^{x+\delta} f(s)\,ds + \frac{1}{f(x)} \int_x^\infty f(u+\delta)\,du \\
&\leq \frac{1}{f(x)} \int_x^{x+\delta} f(s)\,ds + \frac{1}{f(x)} \int_x^\infty \delta f(u)\,du \\
&= \frac{1}{f(x)} \int_x^{x+\delta} f(s)\,ds + \delta F(x).
\end{aligned}$$

Consequently, if $x > M$, then

$$(1-\delta)F(x) \le \tfrac{1}{f(x)} \int_x^{x+\delta} f(s)\,ds \le \tfrac{1}{f(x)} \int_x^{x+\delta} f(x)\,ds = \delta,$$

and so $0 < F(x) \le \frac{\delta}{1-\delta} < \varepsilon$ holds for all $x > M$. Thus, $\lim_{x\to\infty} F(x) = 0$.

Problem 24.6. *Show that the improper Riemann integrals*

$$\int_0^\infty \cos(x^2)\,dx \quad \textit{and} \quad \int_0^\infty \sin(x^2)\,dx$$

(which are known as the **Fresnel integrals***) both exist. Also, show that* $\cos(x^2)$ *and* $\sin(x^2)$ *are not Lebesgue integrable over* $[0,\infty)$.

Solution. We shall work with $\int_0^\infty \sin(x^2)\,dx$. Similar arguments will establish the corresponding result for $\int_0^\infty \cos(x^2)\,dx$.

Let $0 < s < t$. The substitution $u = x^2$ followed by an integration by parts gives

$$\Big|\int_s^t \sin(x^2)\,dx\Big| = \tfrac{1}{2}\Big|\int_{s^2}^{t^2} \tfrac{\sin u}{\sqrt{u}}\,du\Big| = \tfrac{1}{2}\Big|\Big[\tfrac{\cos u}{\sqrt{u}}\Big|_{s^2}^{t^2} - \int_{s^2}^{t^2} \cos u\, d\big(u^{-\frac{1}{2}}\big)\Big]\Big|$$
$$\le \tfrac{1}{2}\Big[\tfrac{1}{s} + \tfrac{1}{t} + \int_{s^2}^{t^2} d\big(u^{-\frac{1}{2}}\big)\Big] = \tfrac{1}{t}.$$

This inequality, combined with Theorem 24.1, guarantees the existence of the improper Riemann integral $\int_0^\infty \sin(x^2)\,dx$. The inequality

$$\int_{\sqrt{k\pi-\pi}}^{\sqrt{k\pi}} |\sin(x^2)|\,dx = \tfrac{1}{2}\int_{k\pi-\pi}^{k\pi} \tfrac{|\sin u|}{\sqrt{u}}\,du \ge \tfrac{1}{2\sqrt{\pi k}} \int_{k\pi-\pi}^{k\pi} |\sin x|\,dx = \tfrac{1}{\sqrt{\pi k}}$$

implies

$$\int_0^{\sqrt{n\pi}} |\sin(x^2)|\,dx = \sum_{k=1}^n \int_{\sqrt{k\pi-\pi}}^{\sqrt{k\pi}} |\sin(x^2)|\,dx \ge \tfrac{1}{\sqrt{\pi}} \sum_{k=1}^n \tfrac{1}{\sqrt{k}},$$

which shows that $\int_0^\infty |\sin(x^2)|\,dx$ does not exist in $\mathbf{R}$—and hence, that $\sin(x^2)$ is not Lebesgue integrable over $[0,\infty)$.

Problem 24.7. *Show that* $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$.

Solution. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{\sin^2 x}{x^2} & \text{if } 0 < x \le 1 \\ \frac{1}{x^2} & \text{if } x > 1, \end{cases}$$

and note that f is Lebesgue integrable over $[0, \infty)$. In view of the inequality $0 \le \frac{\sin^2 x}{x^2} \le f(x)$, we see that the function $\frac{\sin^2 x}{x^2}$ is Lebesgue integrable over $[0, \infty)$.

Now, note that for each r, $\varepsilon > 0$, we have

$$\begin{aligned} \int_\varepsilon^r \frac{\sin^2 x}{x^2}\,dx &= -\int_\varepsilon^r \sin^2 x\, d\left(\tfrac{1}{x}\right) = -\frac{\sin^2 x}{x}\bigg|_\varepsilon^r + \int_\varepsilon^r \frac{2\sin x\cos x}{x}\,dx \\ &= \frac{\sin^2 \varepsilon}{\varepsilon} - \frac{\sin^2 r}{r} + \int_{2\varepsilon}^{2r} \frac{\sin x}{x}\,dx. \end{aligned}$$

Thus, by Theorem 24.8, we see that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \lim_{\substack{r\to\infty \\ \varepsilon\to 0^+}} \int_\varepsilon^r \frac{\sin^2 x}{x^2}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Problem 24.8. *Let $(X, \mathcal{S}, \mu)$ be an arbitrary measure space, T a metric space, and $f: X \times T \to \mathbf{R}$ a function. Assume that $f(\cdot, t)$ is a measurable function for each $t \in T$ and $f(x, \cdot)$ is a continuous function for each $x \in X$. Assume also that there exists an integrable function g such that for each $t \in T$ we have $|f(x,t)| \le g(x)$ for almost all $x \in X$. Show that the function $F: T \to \mathbf{R}$, defined by*

$$F(t) = \int_X f(x,t)\,d\mu(x),$$

is a continuous function.

Solution. Let $t_n \longrightarrow t$ in T. Define the function $g_n: X \longrightarrow \mathbf{R}$ by the formula $g_n(x) = f(x, t_n)$. By our assumptions each g_n is integrable, $|g_n| \le g$ a.e., and $g_n(x) \longrightarrow f(x,t)$ holds for each $x \in X$. Thus, by the Lebesgue Dominated Convergence Theorem, we have

$$F(t_n) = \int f(x, t_n)\,d\mu(x) = \int g_n\,d\mu \longrightarrow \int f(x,t)\,d\mu(x) = F(t).$$

This shows that F is a continuous function.

Problem 24.9. *Show that*

$$\int_0^\infty \frac{e^{-x} - e^{-xt}}{x}\,dx = \ln t$$

holds for each $t > 0$.

Solution. Consider the function $f(x,t) = \frac{e^{-x}-e^{-xt}}{x}$ for $x > 0$ and $t > 0$. Observe that the value $f(0,t) = t-1$ extends f continuously to the point $(0,t)$, $t > 0$. Next, note that the function $g(x,t)$, defined by

$$g(x,t) = \begin{cases} |f(x,t)| & \text{if } 0 \le x \le 1 \text{ and } t > 0 \\ e^{-x} + e^{-xt} & \text{if } x > 1 \text{ and } t > 0\,, \end{cases}$$

is Lebesgue integrable for each $t > 0$. Moreover, $|f(x,t)| \le g(x,t)$ holds. This implies that

$$F(t) = \int_0^\infty f(x,t)\,dx = \int_0^\infty \tfrac{e^{-x}-e^{-xt}}{x}\,dx$$

exists both as an improper Riemann integral and as a Lebesgue integral; see also Theorem 24.3.

Next, note that $\frac{\partial f}{\partial t}(x,t) = e^{-xt}$ holds for all $x > 0$ and all $t > 0$. The inequality $0 \le e^{-xt} \le e^{-xa}$ for all $t > a > 0$ and all $x \ge 0$, coupled with Theorem 24.5, shows that

$$F'(t) = \int_0^\infty \tfrac{\partial f}{\partial t}(x,t)\,dx = \int_0^\infty e^{-xt}\,dx = \tfrac{1}{t}$$

holds for all $t > 0$. Thus, $F(t) = \ln t + C$. Since $F(1) = 0$, it follows that $C = 0$ and so

$$F(t) = \int_0^\infty \tfrac{e^{-x}-e^{-xt}}{x}\,dx = \ln t.$$

Problem 24.10. *For each* $t > 0$, *let* $F(t) = \int_0^\infty \frac{e^{-xt}}{1+x^2}\,dx$.

a. *Show that the integral exists as an improper Riemann integral and as a Lebesgue integral.*
b. *Show that* F *has a second-order derivative and that* $F''(t) + F(t) = \frac{1}{t}$ *holds for each* $t > 0$.

Solution. (a) The integrability of F follows from Theorem 24.3 and the inequality

$$\left|\frac{e^{-xt}}{1+x^2}\right| \le \frac{1}{1+x^2}.$$

(b) If $f(x,t) = \frac{e^{-xt}}{1+x^2}$, then

$$\frac{\partial f}{\partial t}(x,t) = \frac{-xe^{-xt}}{1+x^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial t^2}(x,t) = \frac{x^2e^{-xt}}{1+x^2}.$$

Since $\left|\frac{\partial f}{\partial t}(x,t)\right| \le e^{-xt}$ and $\left|\frac{\partial^2 f}{\partial t^2}(x,t)\right| \le e^{-xt}$ both hold, by applying Theorem 24.5 twice we get

$$F''(t) = \int_0^\infty \frac{\partial^2 f}{\partial t^2}(x,t)\,dx = \int_0^\infty \frac{x^2e^{-xt}}{1+x^2}\,dx.$$

Consequently,

$$F''(t) + F(t) = \int_0^\infty \frac{x^2e^{-xt}}{1+x^2}\,dx + \int_0^\infty \frac{e^{-xt}}{1+x^2}\,dx = \int_0^\infty e^{-xt}\,dx = \frac{1}{t}.$$

Problem 24.11. *Show that the improper Riemann integral $\int_0^{\frac{\pi}{2}} \ln(t\cos x)\,dx$ exists for each $t > 0$ and that it is also a Lebesgue integral. Also, show that*

$$\int_0^{\frac{\pi}{2}} \ln(t\cos x)\,dx = \frac{\pi}{2}\ln\left(\frac{t}{2}\right)$$

holds for all $t > 0$.

Solution. Let $f(x,t) = \ln(t\cos x)$ for $0 \le x < \frac{\pi}{2}$, $t > 0$, and let $g(x) = \left(\frac{\pi}{2} - x\right)^{-\frac{1}{2}}$ for $0 \le x < \frac{\pi}{2}$. An easy argument shows that the improper Riemann integral (and hence, the Lebesgue integral) of g exists over $[0, \frac{\pi}{2})$. Also, L'Hôpital's Rule shows that $\lim_{x\uparrow\frac{\pi}{2}}\left[\frac{f(x,t)}{g(x)}\right] = 0$. Thus, for each $t > 0$ there exists some $0 < x_0 < \frac{\pi}{2}$ such that $|f(x,t)| \le g(x)$ holds for all $x_0 < x < \frac{\pi}{2}$. Since $f(x,t)$ is continuous for $0 \le x < \frac{\pi}{2}$, an easy application of Theorem 22.6 guarantees that

$$F(t) = \int_0^{\frac{\pi}{2}} \ln(t\cos x)\,dx$$

exists both as a Lebesgue and as an improper Riemann integral. Next, note that $\frac{\partial f}{\partial t}(x,t) = \frac{1}{t}$, and that for $0 < a < t$ we have $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq \frac{1}{a}$. Thus, by Theorem 24.5, we have

$$F'(t) = \int_0^{\frac{\pi}{2}} \frac{\partial f}{\partial t}(x,t)\,dx = \frac{\pi}{2t}$$

for each $t > 0$, and therefore $F(t) = \frac{\pi}{2}\ln t + C$.

Since $\int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$ (why?), it follows that

$$\begin{aligned}
2C = 2F(1) &= 2\int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx \\
&= \int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx + \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx \\
&= \int_0^{\frac{\pi}{2}} \ln\left(\tfrac{\sin 2x}{2}\right)dx \\
&= \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,dx - \tfrac{\pi}{2}\ln 2 \\
&= \tfrac{1}{2}\int_0^{\pi} \ln(\sin x)\,dx - \tfrac{\pi}{2}\ln 2 \\
&= C - \tfrac{\pi}{2}\ln 2
\end{aligned}$$

Thus, $C = -\frac{\pi}{2}\ln 2$, and so

$$\int_0^{\frac{\pi}{2}} \ln(t\cos x)\,dx = \tfrac{\pi}{2}\ln t - \tfrac{\pi}{2}\ln 2 = \tfrac{\pi}{2}\ln\left(\tfrac{t}{2}\right)$$

holds for each $t > 0$.

Problem 24.12. *Show that for each $t \geq 0$ the improper Riemann integral $\int_0^\infty \frac{\sin xt}{x(1+x^2)}\,dx$ exists as a Lebesgue integral and that*

$$\int_0^\infty \frac{\sin xt}{x(x^2+1)}\,dx = \tfrac{\pi}{2}(1-e^{-t}).$$

Solution. For each $t \in \mathbb{R}$ let $F(t) = \int_0^\infty \frac{\sin xt}{x(1+x^2)}\,dx$. From

$$\left|\frac{\sin xt}{x(1+x^2)}\right| \leq \frac{|xt|}{|x(1+x^2)|} = \frac{|t|}{1+x^2},$$

we see that F is indeed a well defined real-valued function on $\mathbb{R}$ and that the integral defining F exists both as a Lebesgue integral and as an improper Riemann integral. Moreover, the relations

$$\frac{\partial}{\partial t}\left[\frac{\sin xt}{x(1+x^2)}\right] = \frac{\cos xt}{1+x^2} \quad \text{and} \quad \left|\frac{\cos xt}{1+x^2}\right| \le \frac{1}{1+x^2}$$

in connection with Theorem 24.5 guarantee that F is a differentiable function and

$$F'(t) = \int_0^\infty \frac{\partial}{\partial t}\left[\frac{\sin xt}{x(1+x^2)}\right] dx = \int_0^\infty \frac{\cos xt}{1+x^2}\, dx$$

holds for each $t \in \mathbb{R}$.

Since $\frac{\partial}{\partial t}\left[\frac{\cos xt}{1+x^2}\right] = -\frac{x \sin xt}{1+x^2}$ and the natural dominating function in the inequality $\left|-\frac{x\sin xt}{1+x^2}\right| \le \frac{|x|}{1+x^2}$ is not Lebesgue integrable over $[0, \infty)$, we cannot use Theorem 24.5 to conclude that

$$F''(t) = -\int_0^\infty \frac{x \sin xt}{1+x^2}\, dx. \qquad (\star)$$

As a matter of fact, the identity

$$\frac{x \sin xt}{1+x^2} = \frac{x^2 \sin xt}{x(1+x^2)} = \frac{\sin xt}{x} - \frac{\sin xt}{x(1+x^2)} \qquad (\star\star)$$

shows that, on one hand, the function $x \mapsto \frac{x \sin xt}{1+x^2}$ is not Lebesgue integrable over $[0, \infty)$ for each $t > 0$ and, on the other hand, that

$$\int_0^\infty \frac{x \sin xt}{1+x^2}\, dx = \int_0^\infty \frac{\sin xt}{x}\, dx - \int_0^\infty \frac{\sin xt}{x(1+x^2)}\, dx = \frac{\pi}{2} - F(t) \qquad (\dagger)$$

for each $t > 0$.

We shall establish the validity of $(\star)$ for each $t > 0$ using another method. For each n, let

$$G_n(t) = \int_0^n \frac{\cos xt}{1+x^2}\, dx.$$

Clearly, $G_n(t) \to \int_0^\infty \frac{\cos xt}{1+x^2}\, dx = F'(t)$ for each $t \in \mathbb{R}$. Now, from Theorem 24.5

and (**), we see that

$$G_n'(t) = -\int_0^n \frac{x \sin xt}{1+x^2}\,dx = -\int_0^n \frac{\sin xt}{x}\,dx + \int_0^n \frac{\sin xt}{x(1+x^2)}\,dx$$
$$= -\int_0^{nt} \frac{\sin x}{x}\,dx + \int_0^n \frac{\sin xt}{x(1+x^2)}\,dx,$$

and consequently for each $t > 0$, we have

$$\lim_{n\to\infty} G_n'(t) = -\int_0^\infty \frac{\sin x}{x}\,dx + \int_0^\infty \frac{\sin xt}{x(1+x^2)}\,dx = -\tfrac{\pi}{2} + F(t) = g(t).$$

We claim that for each $a > 0$ the sequence of derivatives $\{G_n'\}$ converges uniformly to the function $g(t) = -\frac{\pi}{2} + F(t)$ on the open interval (a, ∞). To see this, fix $a > 0$ and let $\epsilon > 0$. Choose some $x_0 > 1$ such that

$$\left|\int_s^\infty \frac{\sin x}{x}\,dx\right| < \epsilon \quad \text{and} \quad \left|\int_s^\infty \frac{\sin xt}{x(1+x^2)}\,dx\right| \le \int_s^\infty \frac{dx}{1+x^2} < \epsilon$$

for all $s > x_0$. Now, if we fix some natural number k satisfying $k \ge x_0$ and $ka > x_0$, then for each $n \ge k$ and all $t > a$, we have

$$\left|G_n'(t) - g(t)\right| = \left|\int_{nt}^\infty \frac{\sin x}{x}\,dx - \int_n^\infty \frac{\sin xt}{x(1+x^2)}\,dx\right| < 2\epsilon.$$

This shows that $\{G_n'\}$ converges uniformly to the function $g(t) = -\frac{\pi}{2} + F(t)$.

Finally, using Problem 9.29, we get $F''(t) = \left[\lim_{n\to\infty} G_n(t)\right]' = -\frac{\pi}{2} + F(t)$, or $F''(t) - F(t) = -\frac{\pi}{2}$ for each $t > 0$. Solving the differential equation, we obtain

$$F(t) = \tfrac{\pi}{2} + c_1 e^{-t} + c_2 e^{t} \quad \text{for each } t > 0.$$

Since F and F' are continuous at zero (why?), it follows from $F(0) = 0$ and $F'(0) = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ and the preceding formula of $F(t)$ that $c_1 = -\frac{\pi}{2}$ and $c_2 = 0$. Hence, $F(t) = \frac{\pi}{2}\left(1 - e^{-t}\right)$ for each $t \ge 0$.

Problem 24.13. *The* **Gamma function** *for $t > 0$ is defined by an integral as follows:*

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\,dx.$$

a. *Show that the integral*

$$\int_0^\infty x^{t-1}e^{-x}\,dx = \lim_{\substack{r\to\infty \\ \epsilon\to 0+}} \int_\epsilon^r x^{t-1}e^{-x}\,dx$$

exists as an improper Riemann integral (and hence, as a Lebesgue integral).

b. *Show that* $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

c. *Show that* $\Gamma(t+1) = t\Gamma(t)$ *holds for all* $t > 0$, *and use this conclusion to establish* $\Gamma(n+1) = n!$ *for* $n = 1, 2, \ldots$.

d. *Show that* Γ *is differentiable at every* $t > 0$ *and that*

$$\Gamma'(t) = \int_0^\infty x^{t-1}e^{-x}\ln x\,dx$$

holds.

e. *Show that* Γ *has derivatives of all order and that*

$$\Gamma^{(n)}(t) = \int_0^\infty x^{t-1}e^{-x}(\ln x)^n\,dx$$

holds for $n = 1, 2, \ldots$ *and all* $t > 0$.

Solution. (a) Since $x^{t-1}e^{-x} \le x^{t-1}$ holds for $0 < x \le 1$, it follows from Problem 23.16 that $\int_0^1 x^{t-1}e^{-x}\,dx$ exists both as an improper Riemann integral and as a Lebesgue integral.

Now, for each fixed $t > 0$ we have $\lim_{x\to\infty} x^{t-1}e^{-\frac{x}{2}} = 0$. Thus, there exists some $M > 0$ (depending upon t) such that $0 \le x^{t-1}e^{-\frac{x}{2}} \le M$ holds for all $x \ge 1$. Hence, $x^{t-1}e^{-x} \le Me^{-\frac{x}{2}}$ holds for each $x \ge 1$. This shows that $\int_1^\infty x^{t-1}e^{-x}\,dx$ exists both as an improper Riemann integral and as a Lebesgue integral for each $t > 0$.

The preceding show that $\int_0^\infty x^{t-1}e^{-x}\,dx$ exists both as an improper Riemann integral and as a Lebesgue integral.

(b) Substitute $u = x^{\frac{1}{2}}$ to get

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}}e^{-x}\,dx = 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}.$$

(c) Integrating by parts, we get

$$\begin{aligned}\Gamma(t+1) &= \int_0^\infty x^t e^{-x}\,dx = -\int_0^\infty x^t\,d(e^{-x}) \\ &= -x^t e^{-x}\Big|_0^\infty + \int_0^\infty t x^{t-1}e^{-x}\,dx = t\int_0^\infty x^{t-1}e^{-x}\,dx \\ &= t\Gamma(t).\end{aligned}$$

Consequently, we see that

$$\Gamma(n+1) = n!\Gamma(1) = n!\int_0^\infty e^{-x}\,dx = n!.$$

(d) and (e). Note that $\frac{\partial^n}{\partial t^n}\left(x^{t-1}e^{-x}\right) = x^{t-1}e^{-x}(\ln x)^n$ holds for all $t > 0$ and all $x > 0$.

Now, let $a < t < b$ be fixed and consider the continuous function $h(x,t) = \frac{\partial^n}{\partial t^n}(x^{t-1}e^{-x}) = x^{t-1}e^{-x}(\ln x)^n$, $a < t < b$, $x > 0$. We claim that there exists a Lebesgue integrable function $g\colon (0,\infty) \longrightarrow (0,\infty)$ such that $|h(x,t)| \le g(x)$ holds for all $x > 0$ and all $a < t < b$. If this is the case, then Theorem 24.5 allows us to "*differentiate under the integral sign*," and since $0 < a < b$ are arbitrary this shows that Γ must have derivatives of all orders and that the desired formulas hold. So, we must construct a positive Lebesgue integrable function g over $(0,\infty)$ such that $|h(x,t)| \le g(x)$ holds for each $a < t < b$ and each $x > 0$.

Note that for $x \ge 1$, we have $0 \le x^{t-1} \le x^b$. Using L'Hôpital's rule, we see that

$$\lim_{x\to\infty} x^b e^{-\frac{x}{2}}\left(\ln x\right)^n = \lim_{x\to\infty} \tfrac{x^b}{e^{\frac{x}{4}}} \cdot \lim_{x\to\infty} \tfrac{(\ln x)^n}{e^{\frac{x}{4}}} = 0\cdot 0 = 0,$$

and so there exists some $M > 0$ such that $x^b e^{-\frac{x}{2}}(\ln x)^n \le M$ for all $x \ge 1$. Therefore,

$$\left|h(x,t)\right| \le \left|x^{t-1}e^{-x}\left(\ln x\right)^n\right| \le Me^{-\frac{x}{2}}$$

holds for all $x \ge 1$ and all $a < t < b$.

For the rest of our discussion, we shall need two facts from calculus.

$$\lim_{x\to 0^+} x^a\left(\ln x\right)^n = 0 \quad\text{and}\quad \int_{0^+}^1 x^{a-1}\left(\ln x\right)^n dx = \tfrac{(-1)^n n!}{a^{n+1}}.$$

Both can be proven by induction. For this limit use induction and L'Hôpital's rule by observing that

$$\lim_{x\to 0^+} x^a \ln x = \lim_{x\to 0^+} \tfrac{(\ln x)'}{(x^{-a})'} = \lim_{x\to 0^+} -\tfrac{x^a}{a} = 0$$

and

$$\lim_{x\to 0^+} x^a\left(\ln x\right)^{n+1} = \lim_{x\to 0^+} \tfrac{[(\ln x)^{n+1}]'}{(x^{-a})'} = \tfrac{n+1}{a}\lim_{x\to 0^+} x^a\left(\ln x\right)^n.$$

For the integral, use induction and take into account that

$$\int_{0+}^{1} x^{a-1}\ln x\,dx = \tfrac{1}{a}\int_{0+}^{1}\ln x\,d(x^a) = \tfrac{1}{a}x^a\ln x\Big|_{0+}^{1} - \tfrac{1}{a}\int_{0+}^{1} x^{a-1}\,dx = -\tfrac{1}{a^2}$$

and

$$\begin{aligned}\int_{0+}^{1} x^{a-1}(\ln x)^{n+1}\,dx &= \tfrac{1}{a}\int_{0+}^{1}(\ln x)^{n+1}\,d(x^a)\\ &= \tfrac{1}{a}x^a(\ln x)^{n+1}\Big|_{0+}^{1} - \tfrac{n+1}{a}\int_{0+}^{1} x^{a-1}(\ln x)^n\,dx\\ &= -\tfrac{n+1}{a}\int_{0+}^{1} x^{a-1}(\ln x)^n\,dx.\end{aligned}$$

Since either $(\ln x)^n \geq 0$ holds for all $x \in (0, 1]$ or $(\ln x)^n \leq 0$ holds for all $x \in (0, 1]$, it follows that the function $\phi(x) = x^{a-1}(\ln x)^n, x \in (0, 1]$, is Lebesgue integrable over $(0, 1]$. Now, let

$$g(x) = \begin{cases} x^{a-1}|\ln x|^n & \text{if } 0 < x \leq 1 \\ Me^{-\frac{x}{2}} & \text{if } x \geq 1 \end{cases},$$

and note that g is Lebesgue integrable over $(0, \infty)$. To finish the proof, notice that

$$|h(x,t)| \leq g(x)$$

holds for all $x > 0$ and all $a < t < b$.

Problem 24.14. *Let $f: [0, 1] \to \mathbb{R}$ be a Lebesgue integrable function and define the function $F: [0, 1] \to \mathbb{R}$ by $F(t) = \int_0^1 f(x)\sin(xt)\,d\lambda(x)$.*

a. *Show that the integral defining F exists and that F is a uniformly continuous function.*
b. *Show that F has derivatives of all orders and that*

$$F^{(2n)}(t) = (-1)^n \int_0^1 x^{2n} f(x)\sin(xt)\,d\lambda(x)$$

and

$$F^{(2n-1)}(t) = (-1)^n \int_0^1 x^{2n-1} f(x)\cos(xt)\,d\lambda(x)$$

for $n = 1, 2, \ldots$ and each $t \in [0, 1]$.

c. *Show that $F = 0$ (i.e., $F(t) = 0$ for all $t \in [0, 1]$) if and only if $f = 0$ a.e.*

Solution. (a) Note that for each fixed $t \in [0, 1]$ the function $x \mapsto \sin(xt)$ is continuous and hence, measurable. The inequality $|f(x)\sin(xt)| \le |f(x)|$ guarantees that $x \mapsto f(x)\sin(xt)$ is integrable for each $t \in [0, 1]$. So, F is a well-defined function.

For the uniform continuity of F note that

$$\begin{aligned}
|F(t) - F(s)| &= \left|\int_0^1 f(x)\sin(xt)\,d\lambda(x) - \int_0^1 f(x)\sin(xs)\,d\lambda(x)\right| \\
&= \left|\int_0^1 f(x)\big[\sin(xt) - \sin(xs)\big]\,d\lambda(x)\right| \\
&\le \int_0^1 |f(x)|\,|\sin(xt) - \sin(xs)|\,d\lambda(x) \\
&\le \int_0^1 |f(x)|\,|xt - xs|\,d\lambda(x) \\
&= \left[\int_0^1 |f(x)|\,d\lambda(x)\right]|t - s|
\end{aligned}$$

holds for all $s, t \in [0, 1]$.

(b) Consider the function of two variables $h(x, t) = f(x)\sin(xt)$. Then an easy inductive argument shows that for each $n = 1, 2, \ldots$ we have

$$\frac{\partial^{2n} h(x,t)}{\partial t^{2n}} = (-1)^n x^{2n} f(x)\sin(xt) \quad \text{and} \quad \frac{\partial^{2n-1} h(x,t)}{\partial t^{2n-1}} = (-1)^n x^{2n-1} f(x)\cos(xt)$$

for each $t \in [0, 1]$ and almost all x. This implies $\left|\frac{\partial^n h(x,t)}{\partial t^n}\right| \le |x^n f(x)| = g_n(x)$ for all $t \in [0, 1]$ and almost all x. Since g_n is Lebesgue integrable for each n, it easily follows from Theorem 24.5 that we can "differentiate under the integral sign" and get the desired formulas.

(c) Assume $F(t) = 0$ for each $t \in [0, 1]$. Then $F^{(2n)}(t) = 0$ for all n, and so from (b) we get $\int_0^1 x^{2n} f(x)\sin(xt)\,d\lambda(x) = 0$ for each n and all $t \in [0, 1]$. Letting $t = 1$, we get

$$\int_0^1 x^{2n}\big[f(x)\sin x\big]\,d\lambda(x) = 0$$

for each n. Now, invoke Problem 22.21 to conclude that $f(x)\sin x = 0$ for almost all x. Since $\sin x > 0$ for each $0 < x \le 1$, we easily infer that $f(x) = 0$ for almost all x.

25. APPROXIMATING INTEGRABLE FUNCTIONS

Problem 25.1. *Let $f:\mathbf{R}\to\mathbf{R}$ be a Lebesgue integrable function. Show that*

$$\lim_{t\to\infty}\int f(x)\cos(xt)\,d\lambda(x)=\lim_{t\to\infty}\int f(x)\sin(xt)\,d\lambda(x)=0.$$

Solution. By Theorem 25.2, it suffices to establish the result for the special case $f=\chi_{[a,b)}$. So, let $f=\chi_{[a,b)}$, where $-\infty<a<b<\infty$. In this case, for each $t>0$ we have

$$\begin{aligned}\left|\int f(x)\cos(xt)\,d\lambda(x)\right| &= \left|\int_a^b \cos(xt)\,dx\right|\\ &= \left|\frac{\sin(xt)}{t}\Big|_{x=a}^{x=b}\right| = \left|\frac{\sin(bt)-\sin(at)}{t}\right| \le \frac{2}{t},\end{aligned}$$

and so $\lim_{t\to\infty}\int f(x)\cos(xt)\,d\lambda(x)=0$ holds. In a similar fashion, we can show that $\lim_{t\to\infty}\int f(x)\sin(xt)\,d\lambda(x)=0$.

Problem 25.2. *A function $f:\mathcal{O}\to\mathbf{R}$ (where $\mathcal{O}$ is a nonempty open subset of $\mathbf{R}^n$) is said to be a C^∞-**function** if f has continuous partial derivatives of all orders.*

a. *Consider the function $\rho:\mathbf{R}\to\mathbf{R}$ defined by $\rho(x)=\exp[\frac{1}{x^2-1}]$ if $|x|<1$ and $\rho(x)=0$ if $|x|\ge 1$. Then show that ρ is a C^∞-function such that* $\operatorname{Supp}\rho=[-1,1]$.
b. *For $\epsilon>0$ and $a\in\mathbf{R}$ show that the function $f(x)=\rho(\frac{x-a}{\epsilon})$ is also a C^∞-function with* $\operatorname{Supp} f=[a-\epsilon,a+\epsilon]$.

Solution. (a) We shall establish that $\rho^{(n)}(1)$ exists for each n.

Start by observing that, by L'Hôpital's Rule, $\lim_{t\to\infty}t^ke^{-\frac{1}{2}t}=0$ holds for $k=0,1,2,\ldots$. Notice that if for $0<x<1$ we let $t=\frac{1}{1-x}$, then we have the inequality

$$\left|\frac{x^m e^{\frac{1}{x^2-1}}}{(x^2-1)^k(x-1)}\right|\le\left|\frac{e^{-\frac{1}{2(1-x)}}}{(1-x)^{k+1}}\right|=t^{k+1}e^{-\frac{1}{2}t},$$

from which it follows that

$$\lim_{x\uparrow 1}\frac{x^m e^{\frac{1}{x^2-1}}}{(x^2-1)^k(x-1)}=\lim_{t\uparrow\infty}t^{k+1}e^{-\frac{1}{2}t}=0\quad\text{for } k,m=0,1,2,\ldots. \qquad (\star)$$

Now, by a simple induction argument, we see that for $-1<x<1$ the derivative $\rho^{(n)}(x)$ is a finite sum of terms of the form $\frac{x^m e^{\frac{1}{x^2-1}}}{(x^2-1)^k}$. Using $(\star)$ and another

simple inductive argument, we can also see that $\rho^{(n)}(1) = 0$ holds for $n = 1, 2, \ldots$.
(b) Note that: $f(x) \neq 0$ if and only if $-1 < \frac{x-a}{\varepsilon} < 1$ if and only if $a - \varepsilon < x < a + \varepsilon$. Therefore, $\operatorname{Supp} f = [a - \varepsilon, a + \varepsilon]$.

Problem 25.3. *Let $[a, b]$ be an interval, $\epsilon > 0$ such that $a + \epsilon < b - \epsilon$, and ρ as in the previous exercise. Define $h\colon \mathbf{R} \to \mathbf{R}$ by $h(x) = \int_a^b \rho(\frac{t-x}{\epsilon})\,dt$ for all $x \in \mathbf{R}$. Then show that*

a. *$\operatorname{Supp} h \subseteq [a - \epsilon, b + \epsilon]$,*
b. *$h(x) = c$ (a constant function) for all $x \in [a + \epsilon, b - \epsilon]$,*
c. *h is a C^∞-function and $h^{(n)}(x) = \int_a^b \frac{\partial^n}{\partial x^n} \rho(\frac{t-x}{\epsilon})\,dt$ holds for all $x \in \mathbf{R}$, and*
d. *the C^∞-function $f = h/c$ satisfies $0 \le f(x) \le 1$ for all $x \in \mathbf{R}$, $f(x) = 1$ for all $x \in [a + \epsilon, b - \epsilon]$, and $\int |\chi_{[a,b)} - f|\,d\lambda < 4\epsilon$.*

Solution. For simplicity, let $g_x\colon \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by $g_x(t) = \rho\big(\frac{t-x}{\varepsilon}\big)$, and so $h(x) = \int_a^b g_x(t)\,dt$.
(a) By part (b) of the preceding problem, we know that $\operatorname{Supp} g_x = [x - \varepsilon, x + \varepsilon]$. Thus, if $a < t < b$ and $x \notin [a - \varepsilon, b + \varepsilon]$, then $g_x(t) = \rho\big(\frac{t-x}{\varepsilon}\big) = 0$ (since $\big|\frac{t-x}{\varepsilon}\big| \ge 1$). This implies that $h(x) = 0$ holds for all $x \notin [a - \varepsilon, b + \varepsilon]$, so that $\operatorname{Supp} h \subseteq [a - \varepsilon, b + \varepsilon]$.
(b) If $a + \varepsilon < x < b - \varepsilon$, then $\operatorname{Supp} g_x = [x - \varepsilon, x + \varepsilon]$ and so

$$h(x) = \int_a^b g_x(t)\,dt = \int_{x-\varepsilon}^{x+\varepsilon} \rho\big(\tfrac{t-x}{\varepsilon}\big)\,dt = \varepsilon \int_{-1}^{1} \rho(u)\,du = c > 0.$$

(c) Since every partial derivative $\frac{\partial^n}{\partial x^n} \rho\big(\frac{t-x}{\varepsilon}\big)$ is continuous, it must be bounded on $[a, b]$ (and hence, on $\mathbf{R}$). Now, the desired conclusion follows from Theorem 24.5.
(d) Since $\operatorname{Supp} g_x = [x - \varepsilon, x + \varepsilon]$ and g_x is a positive function, it follows that

$$0 \le h(x) = \int_a^b g_x(t)\,dt = \int_a^b \rho\big(\tfrac{t-x}{\varepsilon}\big)\,dt \le \int_{x-\varepsilon}^{x+\varepsilon} \rho\big(\tfrac{t-x}{\varepsilon}\big)\,dt = c$$

holds for all x. Thus, $f = h/c$ satisfies $0 \le f(x) \le 1$ for all x.
Finally, observe that

$$\big|\chi_{(a,b)} - f\big| \le \chi_{(a-\varepsilon,a+\varepsilon)} + \chi_{(b-\varepsilon,b+\varepsilon)}$$

holds, and so $\int |\chi_{[a,b)} - f|\,d\lambda = \int |\chi_{(a,b)} - f|\,d\lambda < 4\varepsilon$.
The graph of f is shown in Figure 4.3.

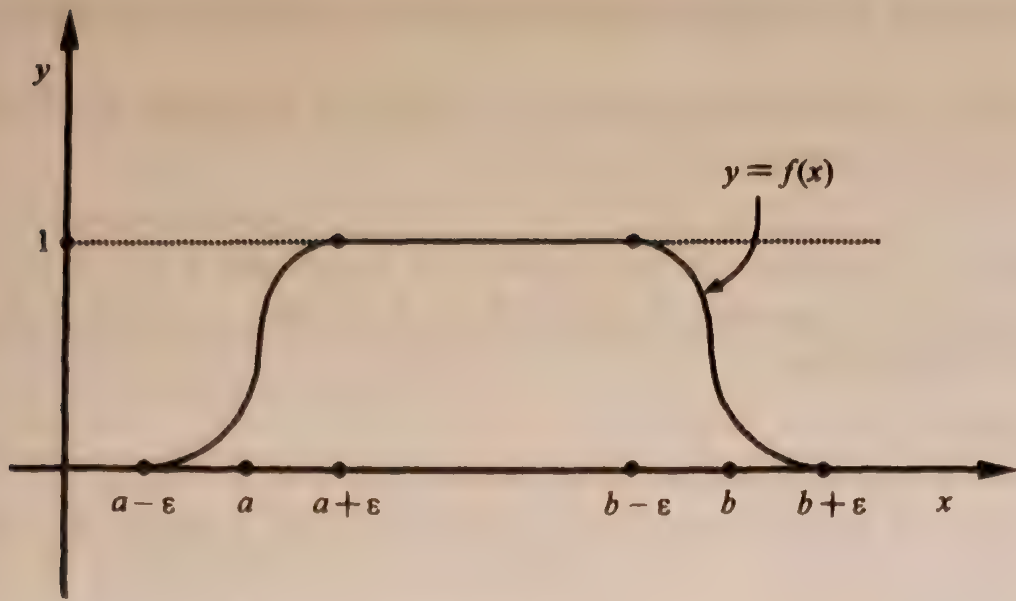

FIGURE 4.3.

Problem 25.4. *Let $f:\mathbb{R} \to \mathbb{R}$ be an integrable function with respect to the Lebesgue measure, and let $\epsilon > 0$. Show that there exists a C^∞-function g such that $\int |f - g| \, d\lambda < \epsilon$.*

Solution. Let $f:\mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue integrable function and let $\varepsilon > 0$. By Theorem 25.2, there is a step function $\phi = \sum_{i=1}^{n} c_i \chi_{[a_i,b_i)}$ (with $c_i \neq 0$ for each i) such that $\int |f - \phi| \, d\lambda < \varepsilon$. Now, by the preceding problem, for each i there exists a C^∞-function g_i with compact support such that $\int |\chi_{[a_i,b_i)} - g_i| \, d\lambda < \frac{\varepsilon}{n|c_i|}$. Now, note that the C^∞-function $g = \sum_{i=1}^{n} c_i g_i$ has compact support and satisfies

$$\begin{aligned}
\int |f - g| \, d\lambda &\leq \int |f - \phi| \, d\lambda + \int |\phi - g| \, d\lambda \\
&< \varepsilon + \int \Big| \sum_{i=1}^{n} c_i \chi_{[a_i,b_i)} - \sum_{i=1}^{n} c_i g_i \Big| \, d\lambda \\
&\leq \varepsilon + \sum_{i=1}^{n} |c_i| \int \big| \chi_{[a_i,b_i)} - g_i \big| \, d\lambda \\
&< \varepsilon + \sum_{i=1}^{n} |c_i| \frac{\varepsilon}{n|c_i|} \\
&= \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}$$

Problem 25.5. *The purpose of this problem is to establish the following general result. If $f:\mathbb{R}^n \to \mathbb{R}$ is an integrable function (with respect to the*

Lebesgue measure) and $\epsilon > 0$, then there exists a C^∞-function g such that $\int|f-g|\,d\lambda < \epsilon$.

a. *Let $a_i < b_i$ for $i = 1, \ldots, n$, and let $I = \prod_{i=1}^n (a_i, b_i)$. Choose $\epsilon > 0$ such that $a_i + \epsilon < b_i - \epsilon$ for each i. Use Problem 25.3 to select for each i a C^∞-function $f_i\colon \mathbf{R} \to \mathbf{R}$ such that $0 \le f_i(x) \le 1$ for all x, $f_i(x) = 1$ if $x \in [a+\epsilon, b_i - \epsilon]$, and $\operatorname{Supp} f_i \subseteq [a_i - \epsilon, b_i + \epsilon]$. Now, define $h\colon \mathbf{R}^n \to \mathbf{R}$ by $h(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i)$. Then show that h is a C^∞-function on $\mathbf{R}^n$ and that*

$$\int |\chi_I - h|\,d\lambda \le 2\Big[\prod_{i=1}^n (b_i - a_i + 2\epsilon) - \prod_{i=1}^n (b_i - a_i)\Big].$$

b. *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be Lebesgue integrable, and let $\epsilon > 0$. Then use part (a) to show that there exists a C^∞-function g with compact support such that*

$$\int |f - g|\,d\lambda < \epsilon.$$

Solution. (a) Clearly, h is a C^∞-function. Let $A = \prod_{i=1}^n (a_i - \varepsilon, b_i + \varepsilon)$, $B = \prod_{i=1}^n (a_i + \varepsilon, b_i + \varepsilon)$, and $C = \prod_{i=1}^n (a_i - \varepsilon, b_i - \varepsilon)$. Now, the desired conclusion follows from the inequality

$$|\chi_I - h| \le (\chi_A - \chi_B) + (\chi_A - \chi_C).$$

(b) Let f be an integrable function and let $\varepsilon > 0$. Pick a step function of the form $\phi = \sum_{i=1}^k c_i \chi_{I_i}$ (where each I_i is a finite open interval of $\mathbf{R}^n$) such that $\int|f - \phi|\,d\lambda < \varepsilon$. From part (a) it follows that there exists a C^∞-function g with compact support such that $\int|\phi - g|\,d\lambda < \varepsilon$. Consequently, $\int|f - g|\,d\lambda < 2\varepsilon$.

Problem 25.6. *Let μ be a regular Borel measure on $\mathbf{R}^n$, f a μ-integrable function, and $\epsilon > 0$. Show that there exists a C^∞-function g such that $\int|f-g|\,d\mu < \epsilon$.*

Solution. Let $I = \prod_{i=1}^n [a_i, b_i]$ be a finite closed interval. Given $\delta > 0$, pick $\varepsilon > 0$ such that the closed interval $J = \prod_{i=1}^n [a_i - 2\varepsilon, b_i + 2\varepsilon]$ satisfies $\mu(J \setminus I) = \mu(J) - \mu(I) < \delta$. (This is always possible since $\prod_{i=1}^n [a_i - \frac{1}{k}, b_i + \frac{1}{k}] \downarrow_k I$.) As in Problem 25.5, there exists a C^∞-function $h\colon \mathbf{R}^n \longrightarrow \mathbf{R}$ such that $0 \le h(x) \le 1$ for all $x \in \mathbf{R}^n$, $h(x) = 1$ for $x \in I$, and $\operatorname{Supp} h \subseteq J$. Therefore, if $f = \chi_I$, then

$$\int |\chi_I - h|\,d\mu = \int (h - \chi_I)\,d\mu \le \int (\chi_J - \chi_I)\,d\mu = \mu(J) - \mu(I) < \delta.$$

Thus, the desired result holds true for the characteristic functions of the finite closed intervals.

Now, let $I = \prod_{i=1}^{n}[a_i, b_i)$ be finite. Since $\prod_{i=1}^{n}[a_i, b_i - \frac{1}{k}] \uparrow_k I$, it follows that the approximation result is also true for the characteristic functions of sets of the form $\prod_{i=1}^{n}[a_i, b_i)$. Since these sets form a semiring and every open set is a σ-set (for this semiring), it is not difficult to see that the result is true for the characteristic functions of open sets of finite measure. The regularity of μ guarantees the validity of the approximation result for characteristic functions of μ-measurable sets of finite μ-measure. This in turn implies that the result holds true for μ-step functions. Finally, since for each μ-integrable function f and each $\varepsilon > 0$, there exists some μ-step function ϕ with $\int |f - \phi|\, d\mu < \varepsilon$, it follows that the C^∞-functions with compact support satisfy the desired approximation property.

Problem 25.7. *Let $f: [a, b] \to \mathbf{R}$ be a Lebesgue integrable function, and let $\epsilon > 0$. Show that there exists a polynomial p such that $\int |f - p|\, d\lambda < \epsilon$, where the integral is considered, of course, over $[a, b]$.*

Solution. Let $f: [a, b] \longrightarrow \mathbf{R}$ be integrable (over $[a, b]$), and let $\varepsilon > 0$. By Theorem 25.3 there exists a continuous function $g: [a, b] \longrightarrow \mathbf{R}$ such that $\int |f - g|\, d\lambda < \varepsilon$. Now, by Corollary 11.6, there exists a polynomial p such that $|g(x) - p(x)| < \varepsilon$ holds for all $x \in [a, b]$. Thus,

$$\int |f - p|\, d\lambda \le \int |f - g|\, d\lambda + \int |g - p|\, d\lambda < \varepsilon + \varepsilon(b - a) = \varepsilon(1 + b - a),$$

and our conclusion follows.

26. PRODUCT MEASURES AND ITERATED INTEGRALS

Problem 26.1. *Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two measure spaces, and assume that $A \times B \in \Lambda_\mu \otimes \Lambda_\nu$.*

a. *Show that $\mu^*(A) \cdot \nu^*(B) \le (\mu \times \nu)^*(A \times B)$.*
b. *Show that if $\mu^*(A) \cdot \nu^*(B) \ne 0$, then $(\mu \times \nu)^*(A \times B) = \mu^*(A) \cdot \nu^*(B)$.*
c. *Give an example for which $(\mu \times \nu)^*(A \times B) \ne \mu^*(A) \cdot \nu^*(B)$.*

Solution. (a) We have $\mathcal{S} \otimes \Sigma \subseteq \Lambda_\mu \otimes \Lambda_\nu$. Let $A \times B \in \Lambda_\mu \otimes \Lambda_\nu$. Also, let $\{A_n \times B_n\}$ be a sequence of $\mathcal{S} \otimes \Sigma$ such that $A \times B \subseteq \bigcup_{n=1}^{\infty} A_n \times B_n$. Since (by Theorem 26.1) $\mu^* \times \nu^*$ is a measure on the semiring $\Lambda_\mu \otimes \Lambda_\nu$, it follows from

Theorem 13.8 that

$$\mu^* \times \nu^*(A \times B) \le \sum_{n=1}^{\infty} \mu^* \times \nu^*(A_n \times B_n) = \sum_{n=1}^{\infty} \mu \times \nu(A_n \times B_n).$$

Consequently, we see that

$$\begin{aligned}
&\mu^*(A) \cdot \nu^*(B) = \mu^* \times \nu^*(A \times B) \\
&\le \inf\Big\{\sum_{n=1}^{\infty} \mu \times \nu(A_n \times B_n) \colon \{A_n \times B_n\} \subseteq \mathcal{S} \otimes \Sigma \text{ and } A \times B \subseteq \bigcup_{n=1}^{\infty} A_n \times B_n\Big\} \\
&= (\mu \times \nu)^*(A \times B).
\end{aligned}$$

(b) If $0 < \mu^*(A) < \infty$ and $0 < \nu^*(B) < \infty$, then

$$(\mu \times \nu)^*(A \times B) = \mu^*(A) \cdot \nu^*(B)$$

holds true by virtue of Theorem 26.2. On the other hand, if either $\mu^*(A) = \infty$ or $\nu^*(B) = \infty$, then—by (a)—the equality holds with both sides equal to ∞.
(c) Let $X = Y = \{0\}$, $\mathcal{S} = \{\emptyset\}$, and $\Sigma = \mathcal{P}(Y)$. Also, let $\mu = 0$ on $\mathcal{S}$ (the only choice!) and $\nu = 0$ on Σ. Now, note that $\mu^*(X) \cdot \nu^*(Y) = \infty \cdot 0 = 0$, while $(\mu \times \nu)^*(X \times Y) = \infty$.

Problem 26.2. *Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two σ-finite measure spaces. Then show that $(\mu \times \nu)^*(A \times B) = \mu^*(A) \cdot \nu^*(B)$ holds for each $A \times B$ in $\Lambda_\mu \otimes \Lambda_\nu$.*

Solution. Let $\{X_n\} \subseteq \Lambda_\mu$ and $\{Y_n\} \subseteq \Lambda_\nu$ satisfy $X_n \uparrow X$, $Y_n \uparrow Y$, $\mu^*(X_n) < \infty$, and $\nu^*(Y_n) < \infty$ for each n. Using Theorems 15.4 and 26.2, we see that

$$\begin{aligned}
(\mu \times \nu)^*(A \times B) &= \lim_{n\to\infty} (\mu \times \nu)^*\big((A \cap X_n) \times (B \cap Y_n)\big) \\
&= \lim_{n\to\infty} \Big[\mu^*(A \cap X_n) \cdot \nu^*(B \cap Y_n)\Big] \\
&= \mu^*(A) \cdot \nu^*(B)
\end{aligned}$$

for each $A \times B \in \Lambda_\mu \otimes \Lambda_\nu$.

Problem 26.3. *Let $(X, \mathcal{S}, \mu)$ and (Y, Σ, ν) be two measure spaces. Assume that A and B are subsets of X and Y, respectively, such that $0 < \mu^*(A) < \infty$, and $0 < \nu^*(B) < \infty$. Then show that $A \times B$ is $\mu \times \nu$-measurable if and only if both A*

and B are measurable in their corresponding spaces. Is the preceding conclusion true if either A or B has measure zero?

Solution. If $A \in \Lambda_\mu$ and $B \in \Lambda_\nu$, then by Theorem 26.3, $A \times B \in \Lambda_{\mu\times\nu}$. For the converse, assume that $A\times B \in \Lambda_{\mu\times\nu}$. We claim first that $(\mu\times\nu)^*(A\times B) < \infty$. To see this, pick two sequences $\{A_n\} \subseteq S$ and $\{B_m\} \subseteq \Sigma$ with $A \subseteq \bigcup_{n=1}^\infty A_n$, $\sum_{n=1}^\infty \mu^*(A_n) < \mu^*(A) + 1$, $B \subseteq \bigcup_{m=1}^\infty B_m$, and $\sum_{m=1}^\infty \nu^*(B_m) < \nu^*(B) + 1$. Now, from $A \times B \subseteq \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty A_n \times B_m$, we see that

$$\begin{aligned}
(\mu \times \nu)^*(A \times B) &\le \sum_{n=1}^\infty \sum_{m=1}^\infty (\mu \times \nu)^*(A_n \times B_m) \\
&= \sum_{n=1}^\infty \sum_{m=1}^\infty \mu(A_n)\cdot\nu(B_m) = \Big[\sum_{n=1}^\infty \mu(A_n)\Big]\cdot\Big[\sum_{m=1}^\infty \nu(B_m)\Big] \\
&< \big[\mu^*(A)+1\big]\cdot\big[\nu^*(B)+1\big] < \infty.
\end{aligned}$$

Therefore, by Theorem 26.4, $(A \times B)^y$ is μ-measurable for ν-almost all y. Since $(A \times B)^y = A$ holds for all $y \in B$ and $\nu^*(B) > 0$, it follows that A is μ-measurable. Similarly, B is ν-measurable.

Finally, if $\mu^*(A) = 0$, $A \neq \emptyset$, and $A \times B \in \Lambda_{\mu\times\nu}$, then B need not be necessarily ν-measurable. An example: Take $X = Y = \mathbf{R}$ with $\mu = \nu = \lambda$. If $E \subseteq [0, 1]$ is nonmeasurable, then $\{0\} \times E$ is a $\mu \times \nu$-null set (since $\{0\} \times E \subseteq \{0\} \times [0, 1]$), and so $\{0\} \times E$ is a $\mu \times \nu$-measurable set.

Problem 26.4. *Let (X, S, μ) and (Y, Σ, ν) be two σ-finite measure spaces, and let $f: X \times Y \to \mathbf{R}$ be a $\mu \times \nu$-measurable function. Show that for μ-almost all x the function f_x is a ν-measurable function. Similarly, show that for ν-almost all y the function f^y is μ-measurable.*

Solution. We can assume $f(x, y) \ge 0$ for each $(x, y) \in X \times Y$. Since (in this case) the product measure is σ-finite, there exists a sequence $\{A_n\}$ of $\mu \times \nu$-measurable sets with $A_n \uparrow X \times Y$ and $(\mu \times \nu)^*(A_n) < \infty$ for each n.

Now, by Fubini's Theorem, the function $(f \wedge \chi_{A_n})_x$ is ν-integrable for μ-almost all x. Since $(f \wedge \chi_{A_n})_x \uparrow f_x$, it follows that f_x is ν-measurable for μ-almost all x.

Problem 26.5. *Show that if $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$, with $f(0, 0) = 0$, then*

$$\int_0^1 \Big[\int_0^1 f(x, y)\,dx\Big]dy = -\frac{\pi}{4} \quad \textit{and} \quad \int_0^1 \Big[\int_0^1 f(x, y)\,dy\Big]dx = \frac{\pi}{4}.$$

Solution. Note that

$$\int_0^1\Big[\int_0^1 f(x,y)\,dx\Big]dy = \int_0^1\Big[-\tfrac{x}{x^2+y^2}\Big|_{x=0}^{x=1}\Big]dy = -\int_0^1 \tfrac{1}{1+y^2}\,dy = -\tfrac{\pi}{4}$$

and

$$\int_0^1\Big[\int_0^1 f(x,y)\,dy\Big]dx = \int_0^1\Big[\tfrac{y}{x^2+y^2}\Big|_{y=0}^{y=1}\Big]dx = \int_0^1 \tfrac{1}{1+x^2}\,dx = \tfrac{\pi}{4}.$$

Problem 26.6. *Let $X = Y = \mathbb{N}$, $S = \Sigma =$ the collection of all subsets of $\mathbb{N}$, and $\mu = \nu =$ the counting measure. Give an interpretation of Fubini's Theorem in this case.*

Solution. Let $f\colon \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{R}$ be a non-negative $\mu\times\nu$-integrable function. Then, by Problem 22.3 and Fubini's Theorem, we see that

$$\int f\,d(\mu\times\nu) = \int\int f\,d\mu d\nu = \int\Big[\sum_{m=1}^{\infty} f(n,m)\Big]d\nu(n) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(n,m).$$

On the other hand, if $f\colon \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{R}$ is a non-negative function such that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(n,m) < \infty,$$

then it follows from Tonelli's Theorem that f is $\mu\times\nu$-integrable. *Conclusion*: A function $f\colon \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{R}$ is $\mu\times\nu$-integrable if and only if $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}|f(n,m)| < \infty$, and in this case

$$\int f\,d(\mu\times\nu) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(n,m) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(n,m).$$

Problem 26.7. *Establish the following result, known as* **Cavalieri's Principle**. *Let (X, S, μ) and (Y, Σ, ν) be two measure spaces, and let E and F be two $\mu\times\nu$-measurable subsets of $X\times Y$ of finite measure. If $\nu^*(E_x) = \mu^*(F_x)$ holds for μ-almost all x, then*

$$(\mu\times\nu)^*(E) = (\mu\times\nu)^*(F).$$

Solution. By Theorem 26.4, we have

$$(\mu \times \nu)^*(E) = \int_X \nu^*(E_x)\, d\mu(x) = \int_X \nu^*(F_x)\, d\mu(x) = (\mu \times \nu)^*(F).$$

Problem 26.8. *For this problem λ denotes the Lebesgue measure on $\mathbf{R}$. Let $(X, \mathcal{S}, \mu)$ be a σ-finite measure space, and let $f: X \to \mathbf{R}$ be a measurable function such that $f(x) \geq 0$ holds for all $x \in X$. Then show that*

a. *The set $A = \{(x, y) \in X \times \mathbf{R}\colon 0 \leq y \leq f(x)\}$, called the* **ordinate set** *of f, is a $\mu \times \lambda$-measurable subset of $X \times \mathbf{R}$.*
b. *The set $B = \{(x, y) \in X \times \mathbf{R}\colon 0 \leq y < f(x)\}$ is a $\mu \times \lambda$-measurable subset of $X \times \mathbf{R}$ and $(\mu \times \lambda)^*(A) = (\mu \times \lambda)^*(B)$ holds.*
c. *The graph of f, i.e., the set $G = \{(x, f(x))\colon x \in X\}$, is a $\mu \times \lambda$-measurable subset of $X \times \mathbf{R}$.*
d. *If f is μ-integrable, then $(\mu \times \lambda)^*(A) = \int f\, d\mu$ holds.*
e. *If f is μ-integrable, then $(\mu \times \lambda)^*(G) = 0$ holds.*

Solution. If $g: X \longrightarrow \mathbf{R}$ is an arbitrary positive measurable function, then we shall write $A_g = \{(x, y) \in X \times \mathbf{R}\colon 0 \leq y \leq g(x)\}$ and

$$B_g = \{(x, y) \in X \times \mathbf{R}\colon 0 \leq y < g(x)\}.$$

Observe that if $f_n(x) \uparrow f(x)$ and $h_n(x) \downarrow f(x)$ hold for each $x \in X$, then $B_{f_n} \uparrow B_f$ and $A_{h_n} \downarrow A_f$.

Assume first that g is a positive simple function. Let $g = \sum_{i=1}^n a_i \chi_{C_i}$ be the standard representation of g, where $a_i > 0$ for each $1 \leq i \leq n$. Then, it is easy to see that

$$A_g = (X \times \{0\}) \cup (C_1 \times [0, a_1]) \cup (C_2 \times [0, a_2]) \cup \cdots \cup (C_n \times [0, a_n]) \quad (\star)$$

and

$$B_g = (C_1 \times [0, a_1]) \cup (C_2 \times [0, a_2]) \cup \cdots \cup (C_n \times [0, a_n]). \quad (\star\star)$$

By Theorem 26.3, both A_g and B_g are $\mu \times \lambda$-measurable subsets of $X \times \mathbf{R}$.

(a) First assume that f is a bounded measurable function. That is, assume that there exists some $M > 0$ such that $0 \leq f(x) \leq M$ holds for all $x \in X$. By Theorem 17.7 there exists a sequence $\{\psi_n\}$ of simple functions with $\psi_n(x) \uparrow M - f(x)$ for each $x \in X$. Thus, the sequence $\{\phi_n\}$ of simple functions, defined by $\phi_n(x) = M - \psi_n(x)$, satisfies $\phi_n(x) \downarrow f(x)$ for each $x \in X$. This implies $A_{\phi_n} \downarrow A_f$. Since (by the preceding discussion) each A_{ϕ_n} is $\mu \times \lambda$-measurable, we see that in this case A_f is likewise a $\mu \times \lambda$-measurable set.

Now, let f be an arbitrary positive measurable function. For each n, let $f_n = f \wedge n\mathbf{1}$, and note that (by the preceding case) each A_{f_n} is $\mu \times \lambda$-measurable. To infer that A_f is a $\mu \times \lambda$-measurable set, observe that $A_{f_n} \uparrow A_f$ holds.
(b) By Theorem 17.7 there exists a sequence $\{s_n\}$ of simple functions such that $0 \le s_n(x) \uparrow f(x)$ holds for all $x \in X$. Clearly, $B_{s_n} \uparrow B_f$ holds. Since each B_{s_n} is $\mu \times \lambda$-measurable, it follows that B_f is likewise $\mu \times \lambda$-measurable.

Next, we shall establish the equality $(\mu \times \lambda)^*(A_f) = (\mu \times \lambda)^*(B_f)$ by cases.

CASE 1. Assume $\mu^*(X) < \infty$.

Clearly, $(\mu \times \lambda)^*(X \times \{0\}) = \mu^*(X) \cdot \lambda(\{0\}) = 0$. Also, assume that $0 \le f(x) \le M < \infty$ holds for all x. Then, there exist two sequences $\{\phi_n\}$ and $\{\psi_n\}$ of step functions with $0 \le \phi_n(x) \uparrow f(x)$ and $\psi_n(x) \downarrow f(x)$ for all $x \in X$. Clearly, $B_{\phi_n} \uparrow B_f$ and $A_{\psi_n} \downarrow A_f$. Now, use ($\star$) and ($\star\star$) in connection with Theorem 26.3 and the Lebesgue Dominated Convergence Theorem to see that

$$\int \phi_n \, d\mu = (\mu \times \lambda)^*(B_{\phi_n}) \uparrow (\mu \times \lambda)^*(B_f) = \int f \, d\mu$$

and

$$\int \psi_n \, d\mu = (\mu \times \lambda)^*(A_{\psi_n}) \downarrow (\mu \times \lambda)^*(A_f) = \int f \, d\mu.$$

Thus, in this case $(\mu \times \lambda)^*(A_f) = (\mu \times \lambda)^*(B_f) = \int f \, d\mu$ holds.

CASE 2. Assume $\mu^*(X) < \infty$ *and that* f *is a positive* μ*-measurable function.*

For each n let $f_n = f \wedge n\mathbf{1}$. Note that $B_{f_n} \uparrow B_f$ and $A_{f_n} \uparrow A_f$. By the preceding case, we have $(\mu \times \lambda)^*(A_{f_n}) = (\mu \times \lambda)^*(B_{f_n})$ for each n. Thus, from Theorem 15.4, it follows that

$$(\mu \times \lambda)^*(A_f) = (\mu \times \lambda)^*(B_f) = \lim_{n\to\infty} \int f_n d\mu.$$

CASE 3. The general case. Here we shall use the hypothesis that μ *is* σ*-finite.*

Choose a sequence $\{E_n\}$ of measurable subsets of X with $E_n \uparrow X$ and $\mu^*(E_n) < \infty$ for each n. Let $g_n = f\chi_{E_n}$, and observe that $B_{g_n} \uparrow B_f$ and $A_{f_n} \uparrow A_f$. Using the preceding case and Theorem 15.4, we see that

$$(\mu \times \lambda)^*(B_f) = \lim_{n\to\infty} (\mu \times \lambda)^*(B_{g_n}) = \lim_{n\to\infty} (\mu \times \lambda)^*(A_{f_n}) = (\mu \times \lambda)^*(A_f).$$

Also, it should be noted here that if f is integrable, then

$$(\mu \times \lambda)^*(B_f) = (\mu \times \lambda)^*(A_f) = \int f\, d\mu.$$

(c) From the identity $G = A_f \setminus B_f$, it follows that the graph G of f is $\mu \times \lambda$-measurable.
(d) The equality follows from the discussion in part (b).
(e) From $G = A_f \setminus B_f$ and part (d), we see that

$$(\mu \times \lambda)^*(G) = (\mu \times \lambda)^*(A_f) - (\mu \times \lambda)^*(B_f) = 0.$$

Problem 26.9. *Let $g\colon X \to \mathbf{R}$ be a μ-integrable function, and let $h\colon Y \to \mathbf{R}$ be a ν-integrable function. Define $f\colon X \times Y \to \mathbf{R}$ by $f(x, y) = g(x)h(y)$ for each x and y. Show that f is $\mu \times \nu$-integrable and that*

$$\int f d(\mu \times \nu) = \Big(\int_X g\, d\mu\Big) \cdot \Big(\int_Y h\, d\nu\Big).$$

Solution. We can assume $g \geq 0$ and $h \geq 0$. Choose a sequence $\{\phi_n\}$ of μ-step functions and a sequence $\{\psi_n\}$ of ν-step functions with $\phi_n \uparrow g$ and $\psi_n \uparrow h$. Then, $\{\phi_n \psi_n\}$ is a sequence of $\mu \times \nu$-step functions such that $\phi_n \psi_n \uparrow gh$. The conclusion now follows from the relation

$$\int \phi_n \psi_n d(\mu \times \nu) = \Big(\int \phi_n\, d\mu\Big) \cdot \Big(\int \psi_n\, d\nu\Big) \uparrow \Big(\int g\, d\mu\Big) \cdot \Big(\int h\, d\nu\Big).$$

Problem 26.10. *Use Tonelli's Theorem to verify that*

$$\int_\epsilon^r \frac{\sin x}{x}\, dx = \int_0^\infty \Big(\int_\epsilon^r e^{-xy} \sin x\, dx\Big)\, dy$$

holds for each $0 < \epsilon < r$. By letting $\epsilon \to 0^+$ and $r \to \infty$ (and justifying your steps) give another proof of the formula

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Solution. Fix $0 < \varepsilon < r$ and consider the function

$$g_{\varepsilon,r}(x,y) = \begin{cases} e^{-xy} & \text{if } (x,y) \in [\varepsilon, r] \times [0,1] \\ e^{-\varepsilon y} & \text{if } (x,y) \in [\varepsilon, r] \times [1, \infty). \end{cases}$$

Clearly, the continuous (and hence, measurable) function $f(x,y) = e^{-xy} \sin x$ satisfies $|f(x,y)| \le g_{\varepsilon,r}(x,y)$ for all $(x,y) \in [\varepsilon, r] \times [0, \infty)$. From

$$\int_\varepsilon^r \Big[\int_1^\infty g_{\varepsilon,r}(x,y)\,dy\Big]\,dx \le \int_\varepsilon^r \frac{1}{\varepsilon}\,dx = \frac{r-\varepsilon}{\varepsilon} < \infty$$

and Tonelli's Theorem, we see that the function $g_{\varepsilon,r}$ is Lebesgue integrable over $[\varepsilon, r] \times [0, \infty)$. So, the function $f(x,y)$ is integrable over $[\varepsilon, r] \times [0, \infty)$. Now Fubini's Theorem guarantees that

$$\int_\varepsilon^r \Big(\int_0^\infty e^{-xy} \sin x\,dy\Big)dx = \int_0^\infty \int_\varepsilon^r \Big(e^{-xy} \sin x\,dx\Big)dy. \qquad (\star)$$

Using the elementary integral

$$\int e^{-\alpha t} \sin t\,dt = -\tfrac{\alpha \sin t + \cos t}{1+\alpha^2}\, e^{-\alpha t}$$

and performing the innermost integrations in $(\star)$, we get

$$\begin{aligned} \int_\varepsilon^r \tfrac{\sin x}{x}\,dx &= \int_0^\infty \Big[-\tfrac{y \sin x + \cos x}{1+y^2}\, e^{-xy} \Big|_{x=\varepsilon}^{x=r}\Big]\,dy \\ &= \int_0^\infty \tfrac{y \sin \varepsilon + \cos \varepsilon}{1+y^2}\, e^{-\varepsilon y}\,dy - \int_0^\infty \tfrac{y \sin r + \cos r}{1+y^2}\, e^{-ry}\,dy, \end{aligned}$$

and consequently,

$$\int_\varepsilon^r \tfrac{\sin x}{x}\,dx = \sin\varepsilon \int_0^\infty \tfrac{ye^{-\varepsilon y}}{1+y^2}\,dy + \cos\varepsilon \int_0^\infty \tfrac{e^{-\varepsilon y}}{1+y^2}\,dy - \int_0^\infty \tfrac{y \sin r + \cos r}{1+y^2}\, e^{-ry}\,dy. \quad (\star\star)$$

We shall compute the limits of the three terms in the right-hand side of $(\star\star)$ as $r \to \infty$ and $\varepsilon \to 0^+$.

We start by computing $\lim_{\varepsilon \to 0^+} \sin\varepsilon \int_0^\infty \frac{ye^{-\varepsilon y}}{1+y^2}\,dy$. To this end, let $\eta > 0$. Since $\lim_{y\to\infty} \frac{y}{1+y^2} = 0$, there exists some $y_0 > 0$ such that $0 < \frac{y}{1+y^2} < \eta$ holds for all $y \ge y_0$. Now, from $\lim_{\varepsilon \to 0^+} y_0 \sin\varepsilon = 0$ and $\lim_{\varepsilon\to 0^+} \frac{\sin\varepsilon}{\varepsilon} = 1$, we see that

there exists some $0 < \delta < 1$ such that

$$0 < \epsilon < \delta \quad \text{implies} \quad 0 < y_0 \sin\varepsilon < \eta \quad \text{and} \quad \tfrac{\sin\varepsilon}{\varepsilon} < 2.$$

Now, if $0 < \varepsilon < \delta$, then (by taking into account that $0 \le \frac{ye^{-\varepsilon y}}{1+y^2} \le 1$ for $y \ge 0$) we infer that

$$\begin{aligned}
\left|\sin\varepsilon \int_0^\infty \tfrac{ye^{-\varepsilon y}}{1+y^2}\,dy\right| &\le \left|\sin\varepsilon \int_0^{y_0} \tfrac{ye^{-\varepsilon y}}{1+y^2}\,dy\right| + \left|\sin\varepsilon \int_{y_0}^\infty \tfrac{ye^{-\varepsilon y}}{1+y^2}\,dy\right| \\
&\le y_0 \sin\varepsilon + \eta\sin\varepsilon \int_{y_0}^\infty e^{-\varepsilon y}\,dy \\
&\le y_0 \sin\varepsilon + \eta\sin\varepsilon \int_0^\infty e^{-\varepsilon y}\,dy \\
&= y_0\sin\varepsilon + \eta\tfrac{\sin\varepsilon}{\varepsilon} < \eta + 2\eta = 3\eta.
\end{aligned}$$

That is, $\lim_{\varepsilon\to 0^+} \sin\varepsilon \int_0^\infty \frac{ye^{-\varepsilon y}}{1+y^2}\,dy = 0$.

For the second limit, note that $\left|\frac{e^{-\varepsilon y}}{1+y^2}\right| \le \frac{1}{1+y^2}$ holds for each $y \in [0,\infty)$. Thus, in view of the Lebesgue integrability of the function $h(y) = \frac{1}{1+y^2}$ over $[0,\infty)$, Theorem 24.4 yields

$$\begin{aligned}
\lim_{\varepsilon\to 0^+}\left[\cos\varepsilon \int_0^\infty \tfrac{e^{-\varepsilon y}}{1+y^2}\,dy\right] &= \lim_{\varepsilon\to 0^+}[\cos\varepsilon]\cdot \lim_{\varepsilon\to 0^+}\left[\int_0^\infty \tfrac{e^{-\varepsilon y}}{1+y^2}\,dy\right] \\
&= 1\cdot \int_0^\infty \lim_{\varepsilon\to 0^+}\left[\tfrac{e^{-\varepsilon y}}{1+y^2}\right]dy = \int_0^\infty \tfrac{dy}{1+y^2} = \tfrac{\pi}{2}.
\end{aligned}$$

For the third limit, note that for each $r \ge 1$ and each $y \ge 0$, we have

$$\left|\tfrac{y\sin r+\cos r}{1+y^2}e^{-ry}\right| \le \tfrac{1+y}{1+y^2}e^{-y} \le 2e^{-y},$$

and so by the Lebesgue integrability of $g(y) = 2e^{-y}$ over $[0,\infty)$, it follows from Theorem 24.4 that

$$\begin{aligned}
\lim_{r\to\infty}\int_0^\infty \tfrac{y\sin r+\cos r}{1+y^2}e^{-ry}\,dy &= \int_0^\infty \lim_{r\to\infty}\left[\tfrac{y\sin r+\cos r}{1+y^2}e^{-ry}\right]dy \\
&= \int_0^\infty 0\,dy = 0.
\end{aligned}$$

Finally, from (★★), we see that

$$\int_0^\infty \tfrac{\sin x}{x}\,dx = \lim_{\substack{\varepsilon\to 0^+\\ r\to\infty}} \int_\varepsilon^r \tfrac{\sin x}{x}\,dx = \lim_{\varepsilon\to 0^+}\Big[\sin\varepsilon \int_0^\infty \tfrac{y}{1+y^2}e^{-\varepsilon y}\,dy\Big] +$$

$$+ \lim_{\varepsilon\to 0^+}\Big[\cos\varepsilon \int_0^\infty \tfrac{e^{-\varepsilon y}}{1+y^2}\,dy\Big] - \lim_{r\to\infty}\int_0^\infty \tfrac{y\sin r+\cos r}{1+y^2}\,e^{-ry}\,dy$$

$$= 0 + \tfrac{\pi}{2} + 0 = \tfrac{\pi}{2}.$$

Problem 26.11. *Show that if* $f(x,y) = ye^{-(1+x^2)y^2}$ *for each x and y, then*

$$\int_0^\infty \Big[\int_0^\infty f(x,y)\,dx\Big]dy = \int_0^\infty \Big[\int_0^\infty f(x,y)\,dy\Big]dx.$$

Use the preceding equality to give an alternate proof of the formula

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Solution. Note that

$$\int_0^\infty \Big[\int_0^\infty ye^{-(1+x^2)y^2}\,dy\Big]dx = \tfrac{1}{2}\int_0^\infty \tfrac{1}{1+x^2}\,dx = \tfrac{\pi}{4}$$

and

$$\int_0^\infty \Big[\int_0^\infty ye^{-(1+x^2)y^2}\,dx\Big]dy = \Big(\int_0^\infty e^{-x^2}\,dx\Big)\cdot\Big(\int_0^\infty e^{-y^2}\,dy\Big)$$
$$= \Big(\int_0^\infty e^{-x^2}\,dx\Big)^2.$$

Since $ye^{-(1+x^2)y^2} \geq 0$ holds for all $x \geq 0$ and $y \geq 0$, Tonelli's Theorem shows that

$$\Big(\int_0^\infty e^{-x^2}\,dx\Big)^2 = \tfrac{\pi}{4}.$$

Problem 26.12. *Show that*

$$\int_0^\infty \Big(\int_0^r e^{-xy^2}\sin x\,dx\Big)dy = \int_0^r \Big(\int_0^\infty e^{-xy^2}\sin x\,dy\Big)dx$$

holds for all $r > 0$. By letting $r \to \infty$ show that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \frac{\sqrt{2\pi}}{2}.$$

In a similar manner show that $\int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{2\pi}/2$.

Solution. Since

$$\begin{aligned}\int_0^\infty\Big(\int_0^r e^{-xy^2}\,dx\Big)\,dy &= \int_0^1\Big(\int_0^r e^{-xy^2}dx\Big)\,dy+\int_1^\infty\Big(\tfrac{1}{y^2}\int_0^{ry^2} e^{-t}dt\Big)dy\\ &\le \int_0^1\Big(\int_0^r 1\,dx\Big)\,dy+\int_1^\infty\Big(\tfrac{1}{y^2}\int_0^\infty e^{-t}\,dt\Big)\,dy\\ &\le r+\int_1^\infty \tfrac{1}{y^2}\,dy = r+1,\end{aligned}$$

it follows from Tonelli's Theorem that e^{-xy^2} is integrable over $[0,r]\times[0,\infty)$. In view of $\left|e^{-xy^2}\sin x\right| \le e^{-xy^2}$, we see that $e^{-xy^2}\sin x$ is also integrable over $[0,r]\times[0,\infty)$, and the stated identity follows from Fubini's Theorem.

Performing the innermost integrations and using the elementary integral

$$\int e^{-\alpha t}\sin t\,dt = -\tfrac{\alpha\sin t+\cos t}{1+\alpha^2}\,e^{-\alpha t},$$

we get

$$\int_0^\infty \tfrac{dy}{1+y^4} - \int_0^\infty \tfrac{y^2\sin r+\cos r}{1+y^4}\,e^{-ry^2}\,dy = \tfrac{\sqrt{\pi}}{2}\int_0^r \tfrac{\sin x}{\sqrt{x}}\,dx. \qquad (\star)$$

Since $\left|\frac{y^2\sin r+\cos r}{1+y^4}e^{-ry^2}\right| \le \frac{1+y^2}{1+y^4} = f(y)$ holds, and f is Lebesgue integrable over $[0,\infty)$, it follows from Theorem 24.4 that

$$\begin{aligned}&\lim_{r\to\infty}\Big[\int_0^r \tfrac{dy}{1+y^4} - \int_0^\infty \tfrac{y^2\sin r+\cos r}{1+y^4}\,e^{-ry^2}\,dy\Big]\\ &= \int_0^\infty \tfrac{dy}{1+y^4} - \lim_{r\to\infty}\Big[\int_0^\infty \tfrac{y^2\sin r+\cos r}{1+y^4}\,e^{-ry^2}\,dy\Big]\\ &= \int_0^\infty \tfrac{dy}{1+y^4} - \int_0^\infty \lim_{r\to\infty}\Big[\tfrac{y^2\sin r+\cos r}{1+y^4}\,e^{-ry^2}\Big]dy\\ &= \int_0^\infty \tfrac{dy}{1+y^4} - \int_0^\infty 0\,dy = \int_0^\infty \tfrac{dy}{1+y^4}.\end{aligned}$$

Now, using the elementary integral

$$\int \frac{dy}{1+y^4} = \frac{1}{4\sqrt{2}}\left[\ln\left(\frac{1+y\sqrt{2}+y^2}{1-y\sqrt{2}+y^2}\right) + 2\arctan\frac{y\sqrt{2}}{1-y^2}\right]$$

and an easy computation, we see that

$$\int_0^\infty \frac{dy}{1+y^4} = \int_0^{1^-} \frac{dy}{1+y^4} + \int_{1^+}^\infty \frac{dy}{1+y^4} = \frac{\pi\sqrt{2}}{4}.$$

Thus, from ($\star$), we see that

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \lim_{r\to\infty}\int_0^r \frac{\sin x}{\sqrt{x}}\,dx = \frac{2}{\sqrt{\pi}}\cdot\frac{\pi\sqrt{2}}{4} = \frac{\sqrt{2\pi}}{2}.$$

Problem 26.13. *Using the conclusions of the preceding problem (and an appropriate change of variable), show that the values of the* **Fresnel integrals** *(see Problem 24.6) are*

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

Solution. Using the change of variable $x = \sqrt{u}$, we get

$$\int_0^r \sin(x^2)\,dx = \frac{1}{2}\int_0^{\sqrt{r}} \frac{\sin u}{\sqrt{u}}\,du \quad\text{and}\quad \int_0^r \cos(x^2)\,dx = \frac{1}{2}\int_0^{\sqrt{r}} \frac{\cos u}{\sqrt{u}}\,du.$$

Now, let $r \to \infty$ and use the preceding problem.

Problem 26.14. *Let $X = Y = [0, 1]$, $\mu =$ the Lebesgue measure on $[0, 1]$, and $\nu =$ the counting measure on $[0, 1]$. Consider the "diagonal" $\Delta = \{(x, x)\colon x \in X\}$ of $X \times Y$. Then show that*

a. *Δ is a $\mu \times \nu$-measurable subset of $X \times Y$, and hence, χ_Δ is a non-negative $\mu \times \nu$-measurable function.*
b. *Both iterated integrals $\int\int \chi_\Delta\,d\mu d\nu$ and $\int\int \chi_\Delta\,d\nu d\mu$ exist.*
c. *The function χ_Δ is not $\mu \times \nu$-integrable. Why doesn't this contradict Tonelli's Theorem?*

Solution. (a) Consider the two sets

$$A = \{(x, y) \in X \times Y\colon x > y\} \quad\text{and}\quad B = \{(x, y) \in X \times Y\colon x < y\}.$$

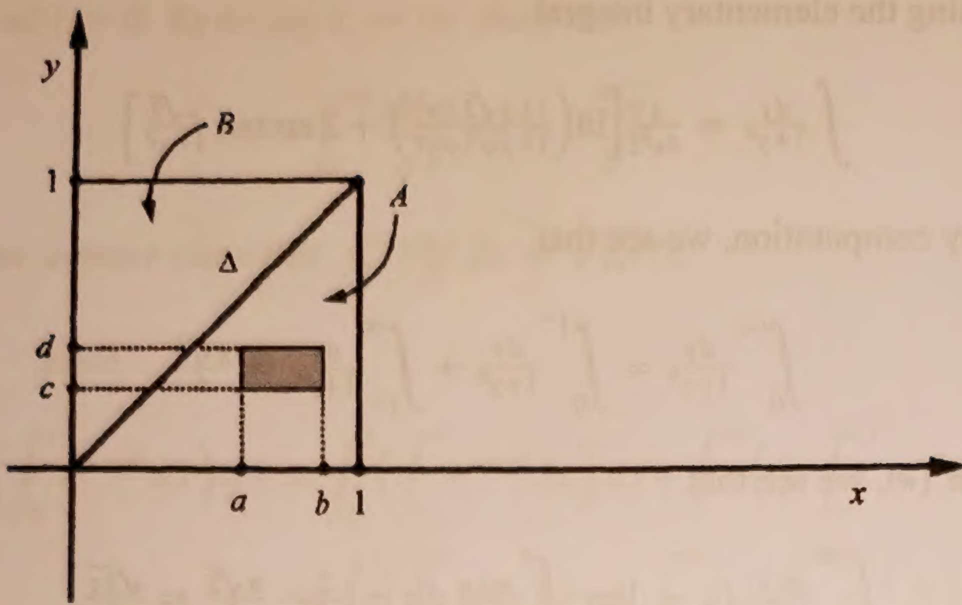

FIGURE 4.4.

Note that $A = \bigcup [a,b] \times [c,d]$, where the union extends over all rectangles $[a,b] \times [c,d]$ with rational end points and $a > d$; see Figure 4.4.

Clearly, the collection of all such rectangles is countable. Since each rectangle is $\mu \times \nu$-measurable, it follows that A is $\mu \times \nu$-measurable. Similarly, the set B is $\mu \times \nu$-measurable. Hence, $\Delta = X \times Y \setminus A \cup B$ is a $\mu \times \nu$-measurable set.

(b) Note that

$$\int\int \chi_\Delta \, d\mu \, d\nu = \int_0^1 \Big[\int_0^1 \chi_\Delta(x,y)\, d\mu(x)\Big] d\nu(y)$$
$$= \int_0^1 \Big[\int_0^1 \chi_{\{y\}} d\mu(x)\Big] d\nu(y) = \int_0^1 0 \cdot d\nu(y) = 0,$$

and

$$\int\int \chi_\Delta \, d\nu \, d\mu = \int_0^1 \Big[\int_0^1 \chi_\Delta(x,y)\, d\nu(y)\Big] d\mu(x)$$
$$= \int_0^1 \Big[\int_0^1 \chi_{\{x\}} \, d\nu(y)\Big] d\mu(x) = \int_0^1 1 \cdot d\mu(x) = 1.$$

(c) Fubini's Theorem combined with part (b) shows that χ_Δ is not integrable over $X \times Y$ (i.e., $(\mu \times \nu)^*(\Delta) = \infty$ must hold). This does not contradict Tonelli's Theorem because ν is not a σ-finite measure.

Problem 26.15. *Let $f: \mathbf{R} \to \mathbf{R}$ be Borel measurable. Then show that the functions $f(x+y)$ and $f(x-y)$ are both $\lambda \times \lambda$-measurable.*

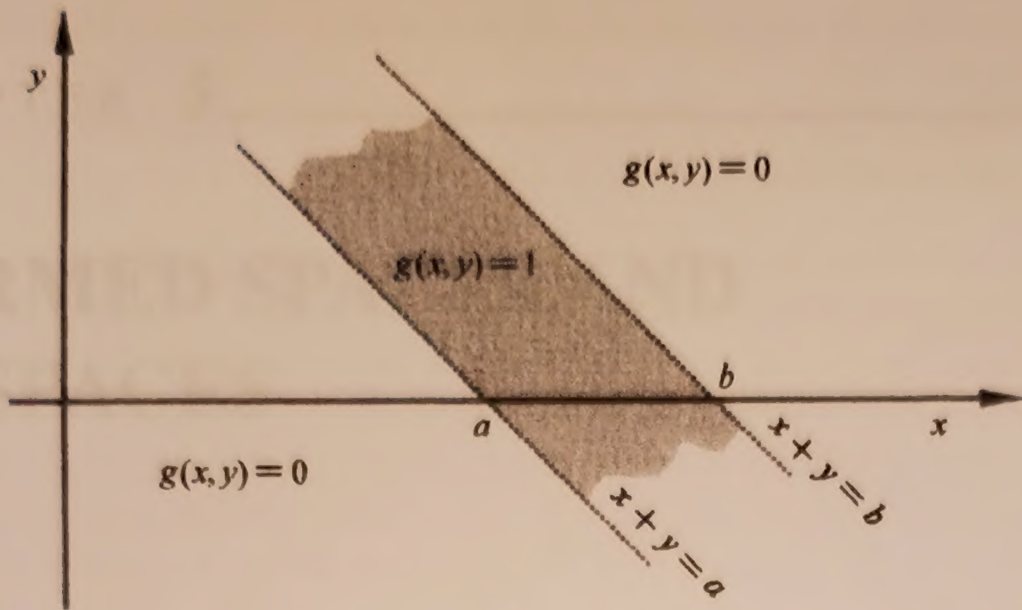

FIGURE 4.5.

Solution. Since f is the limit of a sequence of step functions, it suffices to establish the result for characteristic functions of measurable sets of finite measure. The regularity of the Lebesgue measure allows us to reduce it to the characteristic functions of open sets of finite measure. Finally, this can be reduced to the case when $f = \chi_{(a,b)}$ for some finite open interval (a, b).

The $\lambda \times \lambda$-measurability of the function $g(x, y) = \chi_{(a,b)}(x + y)$ follows easily from the graph shown in Figure 4.5.

The $\lambda \times \lambda$-measurability of $f(x - y)$ can be proven in a similar manner.